JCGM GUM-1:2023

Guide to the expression of uncertainty in measurement — Part 1: Introduction

Guide pour l'expression de l'incertitude de mesure — Partie 1: Introduction

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Joint Committee for Guides in Metrology

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Contents

Foreword

In 1997 a Joint Committee for Guides in Metrology (JCGM), chaired by the Director of the Bureau International des Poids et Mesures (BIPM), was created by the seven international organizations that had originally in 1993 prepared the 'Guide to the expression of uncertainty in measurement' and the 'International vocabulary of basic and general terms in metrology'. The JCGM assumed responsibility for these two documents from the Technical Advisory Group 4 of the International Organization for Standardization (ISO/TAG4).

The Joint Committee is formed by the BIPM with the International Electrotechnical Commission (IEC), the International Federation of Clinical Chemistry and Laboratory Medicine (IFCC), the International Laboratory Accreditation Cooperation (ILAC), the International Organization for Standardization (ISO), the International Union of Pure and Applied Chemistry (IUPAC), the International Union of Pure and Applied Physics (IUPAP) and the International Organization of Legal Metrology (OIML).

JCGM has two Working Groups. Working Group 1, 'Expression of uncertainty in measurement', has the task to promote the use of the 'Guide to the expression of uncertainty in measurement' and to prepare documents for its broad application. Working Group 2, 'Working Group on International vocabulary of basic and general terms in metrology', has the task to revise and promote the use of the 'International vocabulary of basic and general terms in metrology' (the 'VIM').

In 2008 the JCGM made available a slightly revised version (mainly correcting minor errors) of the 'Guide to the expression of uncertainty in measurement', labelling the document 'JCGM 100:2008'.

In 2017 the JCGM rebranded the documents in its portfolio that have been produced by Working Group 1 or are to be developed by that Group. The whole suite of documents is now known as the 'Guide to the expression of uncertainty in measurement' or 'GUM', and is concerned with the evaluation and expression of measurement uncertainty, as well as its application in science, trade, health, safety and other societal activities.

This part of the suite introduces the processes involved and the subsequent parts in this suite giving specific guidance on these processes. This document replaces JCGM 104:2009.

This document has been prepared by Working Group 1 of the JCGM, and has benefited from detailed reviews undertaken by member organizations of the JCGM and National Metrology Institutes.

Guide to the expression of uncertainty in measurement — Part 1: Introduction

1 Scope

The ‘Guide to the expression of uncertainty in measurement’ (GUM) establishes general rules for evaluating and expressing uncertainty in measurement from the shop floor to fundamental research. Therefore, the principles of this suite of documents are intended to be applicable to a broad spectrum of measurements and their applications. An overview of the parts of the GUM is given in table A.1 in Annex A.

NOTE Where the acronym GUM is used in this document, it refers to the suite of documents. An individual part of the GUM is referred to by its corresponding JCGM numbering (e.g., part 6 of the GUM is JCGM GUM-6:2020).

This document gives a rationale for evaluating, expressing and using measurement uncertainty (Clause 2). A brief introduction is given to measurement (Clause 3) and to the decisions involved when evaluating measurement uncertainty (Clause 4). In Clause 5, a brief description of the contents of the parts of the GUM is given. In each of these clauses, the relevant parts of the GUM are identified for further guidance.

2 Rationale

2.1 Measurement [12, Definition 2.1] is one of the most common processes in human activity. Measured values of quantities [12, Definition 2.10] are required for a diverse range of applications and, for each of these values, a statement is needed about its credibility. Such a statement is usually expressed in terms of measurement uncertainty [12, Definition 2.26]. The two components, the measured value and the associated uncertainty, together constitute the most common way of reporting a measurement result [12, Definition 2.9]. In cases where values for more than one quantity are provided by the measurement, a more elaborate statement of the uncertainty is often required (Clause 5.5; see also Clause 3.1).

2.2 Measurements are performed in many branches of society for widely different purposes. Many measurements performed on a daily basis, often in automated processes, concern trade and commerce. Reliable measurement results are also needed for making informed decisions in for example health, safety, weather forecasts, law enforcement and science. Measurements are performed by for example greengrocers, technicians, engineers, laboratory staff, health care professionals, scientists and individuals at home, in very different contexts.

2.3 Many measurements are made for the purpose of comparison of results with specifications. Such a comparison can be made for widely different purposes. This activity is generally known as conformity assessment [3, Definition 4.1] [10, Definition 3.3.1]; see also Clause 5.2.

2.4 Comparison of measurement results is an essential activity in science, calibration and testing. Compatibility of measurement results [12, Definition 2.47] is the basis for being able to reproduce scientific findings, performing quality assurance and quality control, and providing the interpretation of measurement results and informed decision making. Many standards that set requirements for demonstrating competence in measurement require evaluating measurement uncertainty and identifying the major sources of uncertainty, for example ISO/IEC 17025 (calibration and testing laboratories) [4], ISO 17034 (reference material producers) [5], ISO/IEC 17043 (proficiency testing) [6] and ISO 15189 (medical laboratories) [2].

2.5 Measurement uncertainty is of importance in various situations, including but not limited to:

— comparison of measurement results,
— comparison of a measured value with specification limits (conformity assessment),
— establishing metrological traceability [12, Definition 2.41],
— applying a decision rule [10, Definition 3.3.12],
— calculating risks and risk assessment,
— comparison of model outputs and experiments,
— evaluating the validity of models,
— setting limits for the values of physical quantities,
— validating or developing scientific theories, and
— propagation of uncertainty from one measurement to another.

2.6 The GUM substantially contributes to the harmonization of methods for the evaluation, expression and use of measurement uncertainty. It supports the mutual recognition of calibration certificates and laboratory accreditations. The principles and methods of the GUM have been adopted in many documentary standards of ISO, IEC and other standards development organizations. Much software used by laboratories worldwide is based on the provisions of the GUM.

3 Measurement

3.1 The quantity intended to be measured is called the measurand [12, Definition 2.3]. A measurement can have the objective of determining values and associated uncertainties for a set of output quantities, rather than a single quantity. Then the measurand is said to be multivariate. The measurement result is represented by the measured values and the measurement uncertainty.

3.2 Measurement [12, Definition 2.1] can be described as an experimental or computational process that, by comparison with a measurement standard [12, Definition 5.1], produces an estimate of the true value [12, Definition 2.11] of a property, together with a statement of the uncertainty associated with that estimate, and intended for use in support of decision making. This property can be of a material or virtual object or collection of objects, or of a process, event or series of events [19].

NOTE The GUM takes a broad view of 'measurement' in that it recognises that there are instances where the process concerned is essentially computational or where the measurement result is of a conceptual or theoretical nature [7, Clause 1.3].

3.3 In the description of 'measurement' (see Clause 3.2), an estimate [1, Definition 1.31] is an approximation of the true value [12, Definition 2.11] of the property of interest. Another term sometimes used for estimate is measured value [12, Definition 2.10]. By 'comparison with a measurement standard' (see Clause 3.2), it is meant that somewhere in the process a measurement standard is used to obtain an estimate that is metrologically traceable to the relevant measurement unit [12, Definition 1.9].

3.4 Measurement uncertainty is the doubt about the true value of the measurand that remains after making a measurement. Measurement uncertainty can be expressed in various ways. Commonly used ways include:

— a standard uncertainty [12, Definition 2.30],
— an expanded uncertainty [7, Definition 2.3.5] with a coverage factor [7, Definition 2.3.6],
— a coverage interval [12, Definition 2.36] with a stated coverage probability [9, Definition 3.25], or
— a probability distribution describing the knowledge about the measurand [8, Definition 3.1], often expressed as a probability density function [8, Definition 3.3].

3.5 A measurement result should be presented in a way that is understandable and usable by its recipient. The measurement result should therefore include all information needed for its intended use [7, Clause 7]. The information available depends on whether the law of propagation of uncertainty (Clause 5.1), the propagation of distributions (Clause 5.4), or another method [11, Clause 11] has been used for the evaluation of measurement uncertainty.

4 Guidance on evaluating measurement uncertainty

4.1 The evaluation of measurement uncertainty is neither a routine task nor a purely mathematical one. It depends on detailed knowledge of the nature of the measurand and the measurement. The quality and utility of the measurement result therefore ultimately depend on the understanding, critical analysis, and diligence of those who contribute to that result [7, Clause 3.4.8].

4.2 In selecting a method of uncertainty evaluation that suits the current needs, the user should consider the following:

1. the information available,
2. the assumptions to be made,
3. the nature of the results required (see Clause 4.3),
4. the extent to which use is to be made of the information available.

4.3 Alongside the estimate of the measurand, results required from the uncertainty propagation comprise some or all of:

1. standard uncertainty associated with the estimate,
2. coverage interval for the measurand for a stated coverage probability,
3. probability distribution for the measurand.

NOTE 1 The probability distribution for the measurand in Clause 4.3 bullet 3 is the most complete description of the output quantity in terms of the information used. The estimate and the items 1 and 2 can be obtained from it.

NOTE 2 The propagation of distributions (see Clause 5.5) provides the probability distribution for the measurand [8, 9].

NOTE 3 In the case of a multivariate measurand, Clause 4.3 bullets 1 and 2 are generalized: see also Clause 5.5.

4.4 The following information is required for uncertainty propagation:

— A measurement model (mathematical or algorithmic) suitable for the current application, containing input quantities [12, Definition 2.50] of which the user has knowledge and an output quantity (measurand) for which results are required [11],
— Either
    * an estimate of each input quantity – from statistical analysis of observations or provided by other means, and
    * the standard uncertainty associated with each estimate and, when appropriate, the degrees of freedom [1, Definition 2.54] and correlations [1, Definition 2.44] between estimates, or
— a joint probability distribution for the input quantities.

NOTE 1 Guidance on quantifying correlation is given in [7, Clause 5.2] and [11, Clause 10.5].

NOTE 2 Probability distributions for the input quantities are often specified in uncertainty budgets [12, Definition 2.33].

NOTE 3 Guidance on obtaining probability distributions for the input quantities is given in [8, clause 6].

NOTE 4 Guidance on degrees of freedom in the simplest case of uncorrelated repeated observations of an input quantity is given in [7, Annex G] and [9, Clause 6.5.3] for a set of quantities.

4.5 To propagate measurement uncertainty using a measurement model (see also Clause 5.3), it is important to consider what information is available and what is required [11, Clauses 5.1, 5.3, 5.8 and 5.9]. Also the resources necessary to take account of the information are important. Such resources include, for example, human effort, mathematical or similar skills and computational capabilities. Finally, consideration should be given to how the evaluated uncertainty will be used [11, Clauses 12 and 13].

4.6 Taking account of all available knowledge might require the services of a professional statistician, a data scientist or the use of sophisticated software. By contrast, an adequate account of information might only require a spreadsheet and relatively simple calculations. The uncertainty evaluation should be rigorous enough to ensure that measurement results are delivered to the satisfaction of the recipient. Some guidance is given in this document, further elaborated in the other parts of the GUM [7, Clause 3.4] [11, Clauses 6-10], to reach an appropriate balance between effort and outcome.

4.7 Omitting sources that meaningfully contribute to the measurement uncertainty in an uncertainty evaluation should be avoided [11, Clauses 5.6, 7, 9 and 10] [7, Clause 3.4], as it can be detrimental to the compatibility of measurement results, break a metrological traceability chain, or lead to wrong decisions in conformity assessment. Similar considerations apply to using simplifying assumptions that are not justified, such as omitting the evaluation of dependencies between input quantities in a measurement model. Hence, the measurement model should include all sources that meaningfully contribute to the measurement uncertainty and, as appropriate, be supplemented by any dependencies between them.

4.8 The GUM offers two kinds of methods for the evaluation of uncertainty about the values of the input quantities in a measurement model. The choice of method for obtaining an estimate and the associated uncertainty should be based on the information available [7, Clause 4.1]. Guidance is provided for using the statistical analysis of series of data [7, Clauses 4.2, 5, F.1, H.1, H.2, H.3 and H.5] [8, Clause 6.4] [11, clause 11] or using other information [7, Clauses 4.3, 5.2 and F.2] [8, Clause 6]. These statistical methods are known as Type A methods of evaluating standard uncertainty [7, Definition 2.3.2] and the other methods as Type B methods [7, Definition 2.3.3], respectively.

4.9 The GUM offers methods for propagating measurement uncertainty. This propagation involves a measurement model [12, Definition 2.48] (see also [7, Clause 4.1.1]) relating input quantities of which knowledge is available and output quantities about which one wishes to learn. The propagation of the uncertainty associated with the values of the input quantities to the output quantity or quantities depends on their relationships, expressed in the form of the measurement model [7–9, 11]. Even if the measurement considered only consists of taking a reading from an instrument, other factors influencing the measurement result should be taken into account.

4.10 The two principal methods for propagating measurement uncertainty provided in the GUM are the law of propagation of uncertainty (see Clause 5.1) [7, 9, 14–17] and the propagation of distributions by means of the Monte Carlo method of JCGM 101:2008 and JCGM 102:2011 (see Clause 5.4) [8, 9, 18]. The choice of method depends, among other, on the measurement model and the uncertainties about the input quantities [7, Annex G.1] [8, Clause 8.1] [9, Clause 8].

5 Parts of the GUM

5.1 Using the law of propagation of uncertainty (JCGM 100:2008)

5.1.1 JCGM 100:2008 [7] describes the evaluation of measurement uncertainty using a simplified approach. This method takes the estimates and associated standard uncertainties of the input quantities and a linearised form of the measurement model to obtain the estimate and standard uncertainty for the output quantity, the measurand [7, Clause 5]. Based on the assumption that the probability density function for the measurand can be described by a normal distribution or a scaled and shifted t-distribution, a coverage interval corresponding to a specified coverage probability is obtained [7, Clause 6].

5.1.2 JCGM 100:2008 [7] provides general rules for the evaluation and expression of measurement uncertainty [7, Clause 3], intended to be applicable to a broad spectrum of measurements. The guidance in JCGM 100:2008 is mainly concerned with measurands in the form of a well-characterised physical quantity. The method can also be applied in conceptual design and theoretical analysis of experiments, methods of measurement, and complex components and systems [7, Clause 1].

5.1.3 JCGM 100:2008 [7] offers methods for the evaluation of uncertainty about the values of the input quantities in a measurement model. Guidance is provided for using the statistical analysis of series of data [7, Clauses 4.2, 5.2, F.1, H.1, H.2, H.3 and H.5] or other information [7, Clauses 4.3, 5.2 and F.2].

5.1.4 To support the use of JCGM 100:2008 [7], JCGM 101:2008 [8] (see also Clause 5.4) provides a procedure for validating it in any particular case. The procedure is based on a numerical comparison of the results provided by the two approaches, that of JCGM 101:2008 being regarded as a gold standard for the purpose. If these results agree to within the desired numerical accuracy, the application of JCGM 100:2008 can be regarded as acceptable. Otherwise, an alternative approach, such as the propagation of distributions (JCGM 101:2008) itself, should be considered for the uncertainty evaluation [8, Clauses 5.7-5.11].

5.2 Conformity assessment (JCGM 106:2012)

5.2.1 JCGM 106:2012 [10] provides guidance and procedures for assessing the conformity of an item (entity, object or system) to specified requirements. The item might be, for example, a gauge block, a grocery scale, or a blood sample. The procedures can be applied in the following conditions:

— the item is distinguished by a single scalar measurand [10, Clause 1] defined to a sufficient level of detail,
— an interval of permissible values of the property is specified by one or two tolerance limits [10, Definition 3.3.4], and
— the property can be measured and the measurement result [10, Clause 3.2.5] expressed in a manner consistent with the principles of the GUM.

5.2.2 The procedures provided in JCGM 106:2012 [10] can be used to realize an interval, called an acceptance interval [10, Definition 3.3.9 and Clause 8], of permissible measured values of the property of interest. Acceptance limits can be chosen so as to balance the risks associated with accepting non-conforming items (consumer's risk [10, Definitions 3.3.13 and 3.3.15]) and rejecting conforming items (producer's risk [10, Definitions 3.3.14 and 3.3.16]) [10, Clause 9]. Such a choice for the acceptance limits is often part of an agreement between producer and consumer.

5.2.3 Two types of conformity assessment problems are addressed in JCGM 106:2012 [10]. The first is the setting of acceptance limits [10, Clause 8.1] that will assure a desired conformance probability for a single measured item (related to the specific risk). The second is the setting of acceptance limits to assure an acceptable level of confidence on average as a number of (nominally identical) items are measured (related to the global risk). Guidance is given for the solution of both problems.

5.3 Measurement models (JCGM GUM-6)

5.3.1 The measurement model describes the relationship between the input quantities and the measurand [7, Clause 4.1]. It is used for determining the measurement result given the knowledge used. The model should contain every quantity, including all corrections, that contributes to the estimate of the measurand or contributes meaningfully to the uncertainty about the measured value [7, Clause 8] [11, Clause 5.6].

5.3.2 Comprehensive guidance on developing measurement models is given in JCGM GUM-6:2020 [11]. This part of the GUM subdivides the development process leading to a measurement model as follows:

1. specifying the measurand [11, Clause 6],
2. modelling the measurement principle [11, Clause 7],
3. choosing the form of the measurement model [11, Clause 8],
4. identifying effects arising from the measurement [11, Clause 9],
5. adding effects arising from the measurement [11, Clause 10].

5.3.3 One of the challenges in developing measurement models is to identify all quantities to be included. JCGM 100:2008 [7] contains a list of common sources of uncertainty to be considered [7, Clause 3.3.2]. The Eurachem/CITAC guide on quantifying measurement uncertainty [13, Appendix C] contains a customised list for applications in analytical chemistry. Guidance on how to include the quantities in the model is provided in JCGM GUM-6:2020 [11, Clauses 9 and 10] and in the Eurachem/CITAC guide [13].

5.4 Propagation of distributions (JCGM 101:2008)

5.4.1 An evaluation method that can be more widely applied than the law of propagation of uncertainty is the propagation of distributions, often implemented by a Monte Carlo method, as described in JCGM 101:2008 [8]. Rather than taking the estimates and standard uncertainties of the input quantities, this method takes the probability density functions assigned to the input quantities (see Clause 5.1) as input [8, Clause 6]. If some input quantities are correlated, a joint probability density function [1, Definition 2.11 Note 3] is required for these quantities. Using a measurement model, an approximation to the probability density function for the measurand is obtained [8, Clause 7], from which an estimate, standard uncertainty and a coverage interval with the desired coverage probability [8, Clauses 5.3 and 5.5] for the measurand can be calculated. The probability density function thus obtained can also be used in a subsequent uncertainty evaluation.

5.4.2 A general numerical implementation of the propagation of distributions by means of a Monte Carlo method is described in JCGM 101:2008 [8]. With this method, a sample [1, Definition 1.3] of the output probability density function is obtained. The number of trials, i.e., the size of the sample, selected for the Monte Carlo method controls the numerical accuracy of the sample as an approximation to the output probability density function. A procedure by which an appropriate number of trials can be determined is provided [8, Clause 7.9].

5.4.3 For a prescribed coverage probability, JCGM 101:2008 [8] can be used to provide coverage intervals for the output quantity including a probabilistically symmetric and a shortest coverage interval.

5.5 Extension to any number of output quantities (JCGM 102:2011)

5.5.1 JCGM 100:2008 [7] and JCGM 101:2008 [8] focus on measurement models having a single output quantity, i.e., univariate (scalar) models. Many measurement problems arise, however, for which there is more than one output quantity, depending on a common set of input quantities. The models of measurement that apply in such circumstances are referred to as multivariate models [9, Definition 3.8].

5.5.2 To evaluate the uncertainties associated with estimates of the output quantities for multivariate models, the law of propagation of uncertainty and the Monte Carlo method for undertaking the propagation of distributions are generalized. The use of the law of propagation of uncertainty is also generalized to measurement models that are implicit, i.e., in which the input and output quantities are related such that the output quantities cannot be expressed explicitly in terms of the input quantities in a convenient or numerically stable fashion.

5.5.3 JCGM 102:2011 applies the propagation of distributions using the Monte Carlo method from JCGM 101:2008 to multivariate measurement models [9, Clause 7]. Expressions are given for the estimates of the output quantities, and the variances and covariances associated with the estimates, in terms of the numerical output of the Monte Carlo method [9, Clauses 7.4 and 7.6]. Consideration is given, as in JCGM 101:2008, to validating results provided by the (generalized) law of propagation of uncertainty [9, Clause 8], and to the number of trials required by the Monte Carlo method [9, Clause 7.8].

5.5.4 For a prescribed coverage probability, JCGM 102:2011 can be used to provide a coverage region for the output quantities of a multivariate model, the counterpart of a coverage interval for a single scalar output quantity. A procedure for providing an approximation to the smallest coverage region, obtained from results provided by the Monte Carlo method, is also given [9, Clause 7.7].

Annex A Overview of the parts of the GUM

Table A.1: Overview of the parts (published and planned) of the Guide to the expression of uncertainty in measurement

GUM part	Old reference	Title
JCGM GUM-1[a]	JCGM 104	Guide to the expression of uncertainty in measurement — Part 1: Introduction
JCGM GUM-2[b]		Guide to the expression of uncertainty in measurement — Part 2: Concepts
JCGM GUM-3[c]	JCGM 100	Guide to the expression of uncertainty in measurement — Part 3: GUM:1995 with minor corrections
JCGM GUM-4[d]	JCGM 106	Guide to the expression of uncertainty in measurement — Part 4: The role of measurement uncertainty in conformity assessment
JCGM GUM-5[b]	JCGM 110[h]	Guide to the expression of uncertainty in measurement — Part 5: Examples
JCGM GUM-6[e]	JCGM 103[h]	Guide to the expression of uncertainty in measurement — Part 6: Developing and using measurement models
JCGM GUM-7[f]	JCGM 101	Guide to the expression of uncertainty in measurement — Part 7: Propagation of distributions using a Monte Carlo method
JCGM GUM-8[g]	JCGM 102	Guide to the expression of uncertainty in measurement — Part 8: Extension to any number of output quantities
JCGM GUM-9[b]		Guide to the expression of uncertainty in measurement — Part 9: Statistical models and data analysis for interlaboratory studies
JCGM GUM-10[b]		Guide to the expression of uncertainty in measurement — Part 10: Applications of the least squares method
JCGM GUM-11[b]		Guide to the expression of uncertainty in measurement — Part 11: Bayesian methods
JCGM GUM-12[b]		Guide to the expression of uncertainty in measurement — Part 12: Basic methods for uncertainty propagation

[a] JCGM 104:2009 was replaced by this document
[b] Planned
[c] Available as JCGM 100:2008 [7]
[d] Available as JCGM 106:2012 [10]
[e] Available as JCGM GUM-6:2020 [11]
[f] Available as JCGM 101:2008 [8]
[g] Available as JCGM 102:2011 [9]
[h] Reference of the committee draft

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