On logarithmic ratio quantities

The following notes describe how two different logarithmic ratio quantities may be recognised, both of which are dimensionless, but for which the coherent derived SI unit is the neper in one case and the bel in the other case. It seems that this description of the situation accords closely with current practice in the field.

I am writing this paper as background material for the recommendation (CCU-U1-8jun02) which the CCU (the Consultative Committee for Units) plan to submit to the CIPM in October 2002. The CCU is concerned with units, whereas this paper is concerned with the quantities (in this case logarithmic ratio quantities) for which the units neper and bel are used. The definitions, names and symbols for quantities are the province of such bodies as ISO/TC 12, ISO/TC 43 and IEC/TC 29, but the definitions of these quantities are nonetheless relevant to the definitions of the neper and bel.

1. Logarithmic amplitude ratio

The first logarithmic ratio measure of interest might be called **logarithmic amplitude ratio**, or **log arithmic amplitude decay** (or **gain**). This name might be abbreviated to logarithmic decay (or gain), or even just decay (or gain), but in fact the word amplitude is an important part of the name. I have used the symbol μ for this quantity in the equations that follow, for reasons discussed below, but there may be other equally acceptable symbols. This quantity is used *only* for pure sinusoidal signals with exponential decay, represented for example by the equations

$$\mu = \ln(A/A_0) \tag{1}$$

where the signal is denoted by

$$x(t) = A\cos(\omega t), \qquad \text{and} \quad A = A_0 \exp(-\gamma t)\,. \tag{2}$$

ω denotes the angular frequency, t the time, A the amplitude, and γ the decay rate. A_0 is the value of the amplitude A at time $t = 0$. The independent variable may be distance rather than time. The quantity μ is particularly used to represent the exponential decay or gain of a sinusoidal signal with time or distance (decay along a transmission line being an example where the variable is distance). The decay rate γ may be zero as a special case, but the quantity μ may still be used to express the ratio of two amplitudes.

Significant points are:

- Logarithmic amplitude ratio is a dimensionless quantity, and it is always defined using a natural (or Napierian) logarithm (in these notes ln denotes $\log_e$).
- Its coherent derived unit is one, but it is given the special name neper, symbol Np, so that we write

$$\mu = \ln(A/A_0)\ \text{Np}\,. \tag{3}$$

- The name 'level', and symbol L, is sometimes used for this quantity, but I understand that they are not used by those working in the field. In these notes I have reserved the name 'level' and symbol L for the quantities discussed in Section 2.

All of this is in accord with the paper by Mills, Taylor and Thor [1], except for the revised name of the quantity, and the use of the symbol μ rather than L. When equation (2) is written in the form

$$x(t) = \mathrm{Re}[A_0 \exp(-\gamma t + \mathrm{i}\omega t)] \tag{4}$$

we see the parallel relation between the units neper (for the logarithmic amplitude decay γt) and radian (for the phase angle ωt). This relation between the neper and the radian is emphasised in the paper by Mills et al. [1].

Equation (4) implies that $x(t)$ is a generalized single-frequency signal, with complex frequency $(\omega+\mathrm{i}\gamma)$. It is important that the logarithmic amplitude ratio be limited to signals of this type.

2. Mean square signal level and power level

The other logarithmic ratio measures of interest are those formed from the various power-like quantities that commonly arise in acoustics, and also in other applications such as signal processing and telecommunications. They fall into two groups.

One group should really be called the **mean square signal level**, for example the **mean square sound pressure level** in acoustics. The words *mean square* are often omitted in this name, so that the quantity is simply called the **signal level** or **sound pressure level**, but the average of the square of the signal over a suitably chosen time window is always implied. These quantities are defined by the equations

$$L_X = \lg(X/X_0)\,, \tag{5}$$

where $$X = \langle p^2 \rangle. \tag{6}$$

The quantity $p(t)$ is the sound pressure as a function of time in acoustics, or the signal (for example voltage) as a function of time in other applications. The conventional reference value X_0 in acoustics is taken to be $(20\ \mu\mathrm{Pa})^2$.

The other group consists in acoustics of the **mean sound power level**, L_P, the **mean sound intensity level**, L_I, and the **sound exposure level**, L_E**.** The quantities sound power, sound intensity, and sound exposure are inherently quadratic in nature, so there is no reference to 'square' in their name, as there was in the first group. However, the mean power and intensity are still defined as averages over a specified time interval. In practice the adjective 'mean' is commonly omitted in these two names, which thus become **sound power level** and **sound intensity level**. Note that **sound exposure** is defined as an integral of the squared pressure over time, and no further time averaging is required.

The defining equations equivalent to (5) and (6) are:

$$L_P = \lg(P/P_0) \tag{7}$$

$$L_I = \lg(I/I_0) \tag{8}$$

$$L_E = \lg(E/E_0) \tag{9}$$

where in these equations P and I denote the mean power and mean intensity, and E denotes the sound exposure, in each case defined over an appropriate time interval. The conventional reference values for these quantities are $P_0 = 10^{-12}$ W, $I_0 = 10^{-12}$ W/m^2, and $E_0 = 1$ Pa^2 s.

All the quantities (5) through (9) share the general characteristic that their underlying signals typically contain many different frequency components, rather than being confined to a single frequency. It is thus not possible to define an amplitude associated with any of these quantities, which are not repeating functions. The name *level* and the symbol L are appropriate and are commonly used for all these quantities. The reference value, denoted with a subscript 0, must be specified to give meaning to the level L for any of these quantities (although it is often taken by convention as stated above, and may not be explicitly mentioned). In some applications the signal is 'time-stationary', so that the mean square value averaged over a suitable time interval is the same whenever it is taken; in other applications the signal is a decaying (or fluctuating) function of time, so that as the time window over which the average is taken moves forwards in time the mean square value changes. But for decaying free oscillations from some initial excited state, for example the sound field in a reverberant room, the decrease will not generally be exponential, because different Fourier components decay at different rates.

- It is customary to use a decimal logarithm in the defining equations (5), (7), (8) and (9). (In these notes lg denotes $\log_{10}$.) The level L is always a dimensionless quantity, and when it is defined as in equations (5) through (9) its coherent derived unit is one, but it is given the special name bel, symbol B. However its submultiple the decibel, dB, is much more commonly used, so that we may write

  $L_X = \lg(X/X_0)\ \mathrm{B} = 10\lg(X/X_0)\ \mathrm{dB}$ (10)

  with similar equations for L_P, L_I and L_E in (7) through (9).

- There is no relation between the bel and the radian analogous to that between the neper (for logarithmic amplitude decay) and the radian (for phase angle). Thus the analogy exists between the radian and the neper, but there is no corresponding analogy between the radian and the bel. This is because there is no amplitude that can be defined for these functions, as there is for logarithmic amplitude ratio in equation 2.

3. Relation between the neper and the bel

Since the neper and the bel (or decibel) are used for two different kinds of quantity, one can only obtain a relation between these units by relating the quantities. In practice, however, as noted above it is not generally possible to define a sinusoidal signal amplitude associated with a mean square signal level, mean square sound pressure level or mean power level, etc., so that in general L and μ cannot be related. The only exception occurs for a pure sinusoidal signal, for which the mean square signal X corresponds to half the square of the amplitude A. In this case we may average the square of $x(t) = A\cos(\omega t)$ to obtain $\langle x^2\rangle = (1/2)\langle A^2\rangle$, because $\langle\cos^2(\omega t)\rangle = 1/2$. Thus in this special case we arrive at the relation

$$X = (1/2)A^2 \quad \text{and hence} \quad X/X_0 = (A/A_0)^2 \tag{11}$$

where I have written A^2 for $\langle A^2\rangle$. This leads to the relation between L_X and μ:

$$L_X = \lg(X/X_0) = 2\lg(A/A_0) = (2/\ln 10)\ln(A/A_0) = (2/\ln 10)\,\mu \tag{12}$$

Thus if $L_X = 1\ \mathrm{B}$, then $\mu = (\ln 10)/2\ \mathrm{Np} = 1.151\,293\ \mathrm{Np}$ (13)

This is sometimes described by saying that 1 B = (ln 10)/2 Np, but since the bel and the neper are units of different quantities it is perhaps better to express the relation as in (13).

4. Comment

It is a rule of the International System that for any given quantity there is only one coherent SI unit. It may seem that in the procedure described in sections 1 and 2 we have defined two different coherent units, the neper and the bel, for the same quantity. Although both quantities are dimensionless, and both are defined as the logarithm of a ratio, the differences described in Sections 1 and 2 support the argument that they are different quantities. They are normally used in different circumstances. They can only be related in the special case described in section 3.

There are precedents for recognising two slightly different quantities, as suggested here, despite the fact that they are defined in a somewhat similar way. Thus for example angular velocity, $d\theta/dt$, and frequency, f or ν, are commonly regarded as different quantities, each having a different coherent derived SI unit, radian per second (rad/s) for the former and hertz ($Hz = s^{-1}$) for the latter. Yet it is sometimes convenient to relate these two quantities by the equation

$$d\theta/dt = 2\pi f \tag{14}$$

Some would argue that angular velocity and frequency are really the same quantity, and that we have contrived to break the rule by having two different coherent derived units for the same quantity. Underlying this are two different ways of defining plane angle, one leading to the radian as the unit and the other to the revolution. We have lived with this situation for many years, and it does not seem to cause problems. We find it convenient to recognise the two different quantities, each with its own coherent derived SI unit. Similarly in the case of logarithmic ratio, I suggest in this note that it is convenient and customary to recognise two different quantities as described in sections 1 and 2, each with its own coherent derived unit, the neper and the bel. It seems to me that with this approach we might reach agreement between all parties with an interest in this subject.

It is important to use names that clearly distinguish the two different kinds of quantity described in sections 1 and 2. In these notes I am using the name **logarithmic amplitude ratio** for the quantity described in section 1, which is characterised by a single frequency, and for which the coherent unit is the neper. I am using the name **level**, or **mean square signal level**, or **mean power level** etc. for the quantities described in section 2, for which there is no single frequency and for which it is not possible to define an amplitude, and for which the coherent unit is the bel. This is in accord with current practice in so far as I have been able to determine.

Ian Mills, 8 June 2002

[1] I M Mills, B N Taylor and A J Thor, *Metrologia* 2001, **38**, 353 – 361.

Appendix:

On the mathematical definition of the quantities for which the neper and the bel are units

It is sometimes argued that the radian may be understood as a dimensionless derived unit, in contrast to the neper, because the angle θ of which the radian is a unit may be defined as the ratio of two lengths. The lack of a similar definition of the quantity of which the neper is a unit is then said to inhibit its interpretation as a dimensionless derived unit.

This appendix is added to demonstrate that the radian and the neper may both be described as having analogous geometrical interpretations.

The function in equation (4) is of the type

$$F = r \exp(\mathrm{i}\theta + \varphi)$$

If we define a circle by the parametric equations

$$x = r \cos \theta$$

$$y = r \sin \theta$$

where $r \exp(\mathrm{i}\theta) = r \cos\theta + \mathrm{i}\, r \sin \theta$,

then θ may be defined as the (shaded area)/r^2 in the diagram in figure 1.

Similarly if we define a hyperbola by the parametric equations involving the hyperbolic functions

$$x = r \cosh\varphi$$

$$y = r \sinh \varphi$$

where $r \exp \varphi = r \cosh \varphi + r \sinh \varphi$,

then φ may be defined as the (shaded area)/r^2 in the diagram in figure 2.

The figures are attached as a .pdf file.

Thus the both of the quantities θ and φ, which in our application are the phase angle and the logarithmic amplitude decay, may be given a geometrical interpretation in terms of the dimensionless ratio of two areas, related to a circle and a hyperbola respectively. When θ and φ are defined in this way, the radian is the value of θ when $\theta = 1$, and the neper is the value of φ when $\varphi = 1$. This geometrical argument may help to support the picture that the radian and the neper may both be interpreted as dimensionless derived units.

I am indebted to Christian Bordé and Jean Kovalevski for drawing my attention to this geometrical interpretation of the radian and the neper.

Figure 1. If the circle is defined parametrically by the equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

where r is the radius of the circle, and the angle θ is measured in radians, then the area of the circle is πr^2. If the angle between the two bounding radii of the shaded area is 2θ, then the shaded area is θr^2. Thus the (shaded area)/$r^2 = \theta$.

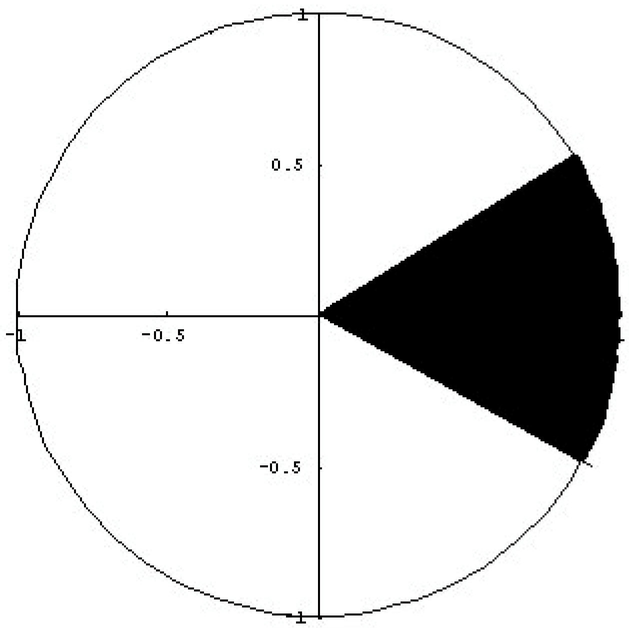

Figure 2. If the hyperbola is defined parametrically by the equations

$$x = r \cosh \varphi , \quad y = r \sinh \varphi$$

so that the distance from the origin to the nearest point on the hyperbola is r, then it may be shown that the shaded area is equal to φr^2. Thus the (shaded area)/$r^2 = \varphi$.