A check of consistency of available results concerning the Planck constant

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CCM Recommendation G1 (2013) On a new definition of the kilogram

1. **At least three independent experiments, including work from watt balance and XRCD experiments, yield consistent values of the Planck constant with relative standard uncertainties not larger than 5 parts in 10^8,**

2. **At least one of these results should have a relative standard uncertainty not larger than 2 parts in 10^8,**

What is ‘independence’?

• Two quantities are said to be *independent* if information about one quantity is completely irrelevant for the other quantity and *vice versa*. Otherwise, they are said to be *dependent*, in which case the joint probability density function (PDF) of the corresponding random variables and further parameters, that is, covariances, must be considered. (JCGM 100:201X, 7.4.2)

• Reference to quantities or estimates being independent or correlated, although used for brevity in the *Guide*, is informal since independence and correlation strictly relate to the corresponding random variables. (*ibid.*, 6.5)

What is ‘consistency’?

- No guidance in the GUM

- Loosely speaking, a data set is consistent when the data scattering is comparable to the individual declared uncertainties

- On a more rigorous footing, a data set is consistent when it satisfies a *consistency criterion*.

- χ^2 (chi squared)

χ^2 (chi squared) distribution

The random variable (RV) χ_ν^2 is the sum of the squares of ν independent RVs X_i having a standard normal distribution $X_i \sim \mathrm{N}(0, 1)$

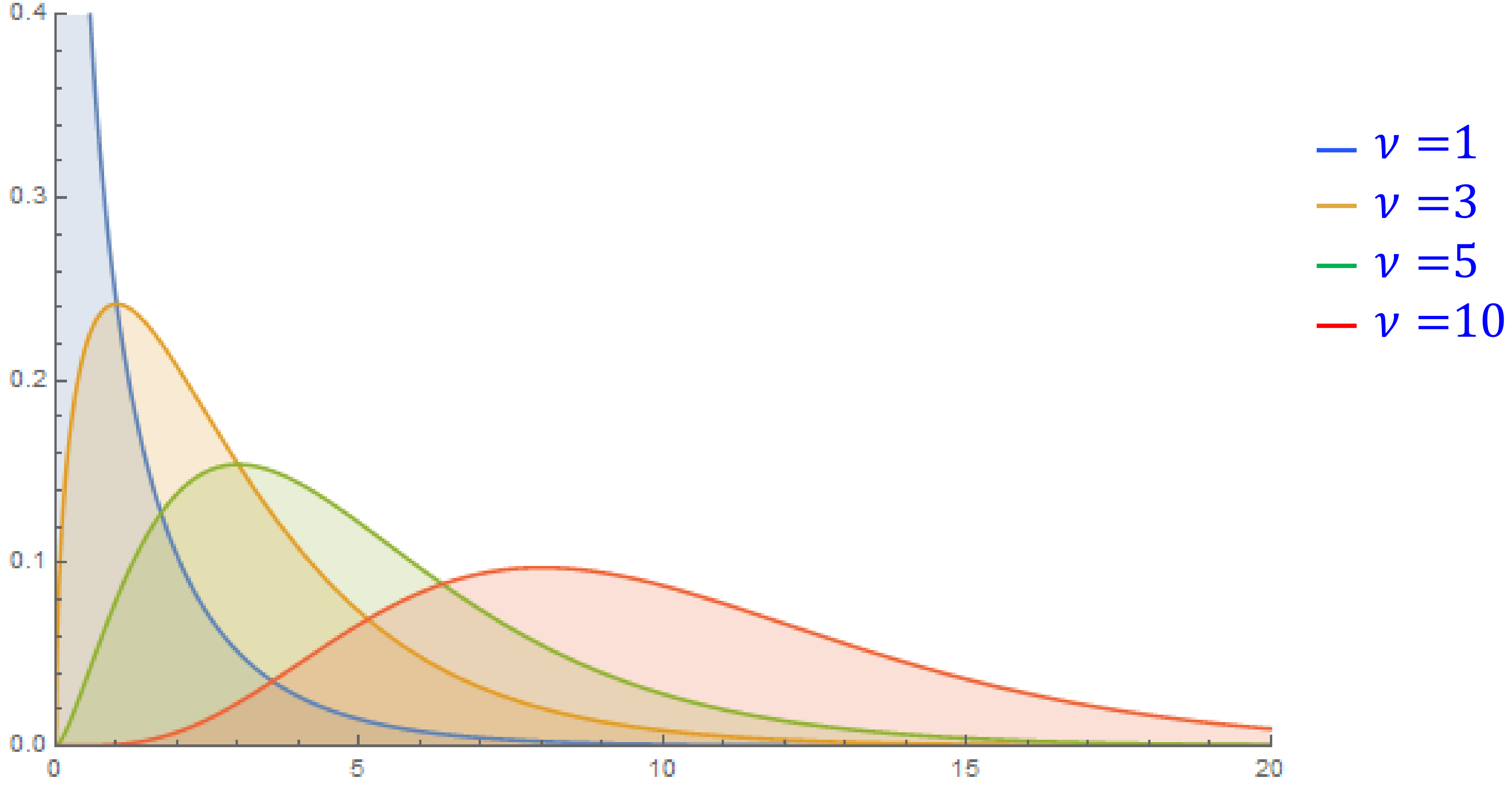
$$\chi_\nu^2 = \sum_{i=1}^{\nu} X_i^2$$

with probability density function (PDF)

$$\chi_\nu^2 \sim \begin{cases} \dfrac{2^{-\nu/2}\, e^{-\nu/2}\, x^{-1+\nu/2}}{\Gamma\left[\frac{\nu}{2}\right]} & x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

$E(\chi_\nu^2) = \nu$ and $V(\chi_\nu^2) = 2\nu$

χ^2 probability density function (PDF)



Weighted mean

Given a set of N independent values x_i **presumed to estimate the same quantity** μ, and the associated uncertainties $u(x_i)$, the popular weighted mean

$$\hat{\mu} = \frac{\dfrac{x_1}{u^2(x_1)} + \cdots + \dfrac{x_N}{u^2(x_N)}}{\dfrac{1}{u^2(x_1)} + \cdots + \dfrac{1}{u^2(x_N)}}$$

is the best linear estimator for μ, with (squared) uncertainty

$$u^2(\hat{\mu}) = \frac{1}{\dfrac{1}{u^2(x_1)} + \cdots + \dfrac{1}{u^2(x_N)}}$$

Weighted mean

If covariances are meaningful, the generalised (matrix) expression is

$$\hat{\mu} = u^2(\hat{\mu})\, \mathbf{1}^\top\, \boldsymbol{U}(\boldsymbol{x})^{-1}\, \boldsymbol{x}$$

with

$$u^2(\hat{\mu}) = [\ \mathbf{1}^\top\, \boldsymbol{U}(\boldsymbol{x})^{-1} \mathbf{1}]^{-1}$$

Here $\mathbf{1} = (1, 1, \cdots, 1)^\top_{N\times 1}$ and $\boldsymbol{U}(\boldsymbol{x})$ is the covariance matrix

Weighted mean

The WM provides reliable results only if the data scattering is purely random, i.e, if the associated RVs X_i are distributed as $X_i \sim \mathrm{N}\left(\mu, \sigma_i^2\right)$

If this is the case, it happens that

$$\sum_{i=1}^{N} \frac{(X_i - \hat{\mu})^2}{\sigma_i^2} \sim \chi_\nu^2$$

Or, if covariances are non-zero

$$(\boldsymbol{X} - \mathbf{1}\hat{\mu})^\top \boldsymbol{V}(\boldsymbol{X})^{-1}(\boldsymbol{X} - \mathbf{1}\hat{\mu}) \sim \chi_\nu^2$$

where $\nu = N - 1$ is the degrees of freedom

χ^2 criterion

To check that the weighted mean can safely be used, the statistic

$$\chi^2_{\mathrm{obs}} = \sum_{i=1}^{N} \frac{(x_i - \hat{\mu})^2}{u^2(x_i)}$$

or, if covariances are non-zero

$$\chi^2_{\mathrm{obs}} = (\boldsymbol{x} - \boldsymbol{1}\hat{\mu})^\top \, \boldsymbol{U}(\boldsymbol{x})^{-1} (\boldsymbol{x} - \boldsymbol{1}\hat{\mu})$$

is formed and checked against χ^2_ν, by calculating

χ^2 criterion

$$p = \Pr\{\chi_\nu^2 > \chi_{\text{obs}}^2\},$$

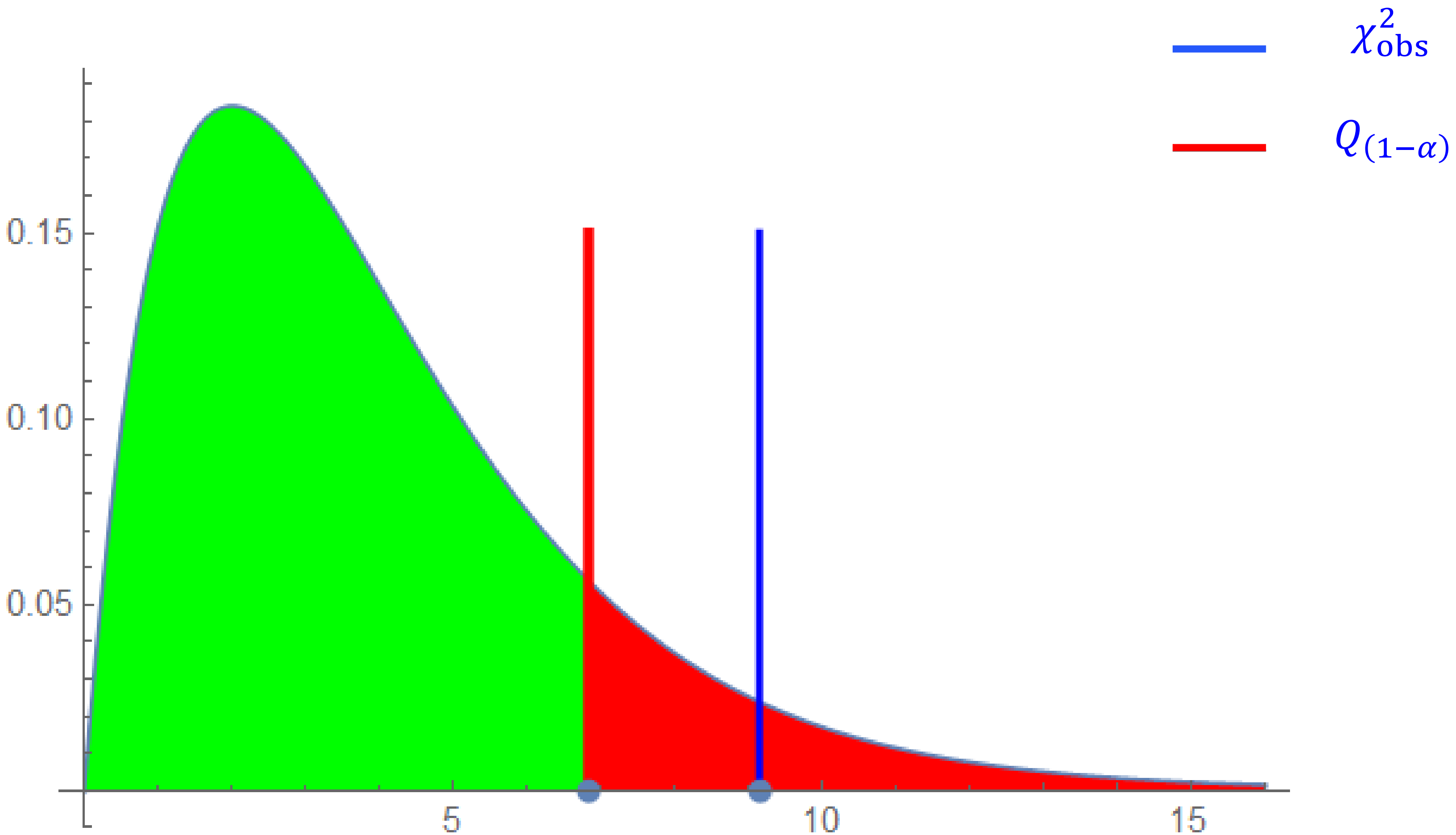
and requesting that $p > \alpha,$

where α is a probability suitably chosen.

p is the probability to obtain a statistic equal to or larger than χ_{obs}^2 if the data scattering is due to purely random effects (a condition for the safe application of the weighted mean).

(For a continuous distribution, the probability of a value is zero!)

χ^2 criterion



Caveats

A different, potentially misleading way of saying is that 'data is consistent at the $100(1-\alpha)$ % confidence level'.

It conveys the false idea that the higher is $100(1-\alpha)$ %, the higher is the confidence that data are random. Things go the other way round!

$100(1-\alpha)$ % is **not** the probability that data is consistent given that χ^2_{obs} passes the test.

Rather, α is the probability of being wrong in rejecting consistency!

Choice of α

The choice of the level of significance α depends on the application.

A possible choice is $\alpha = 0.05$, which means that any χ^2_{obs} lying within the 95th percentile implies acceptance of data consistency.

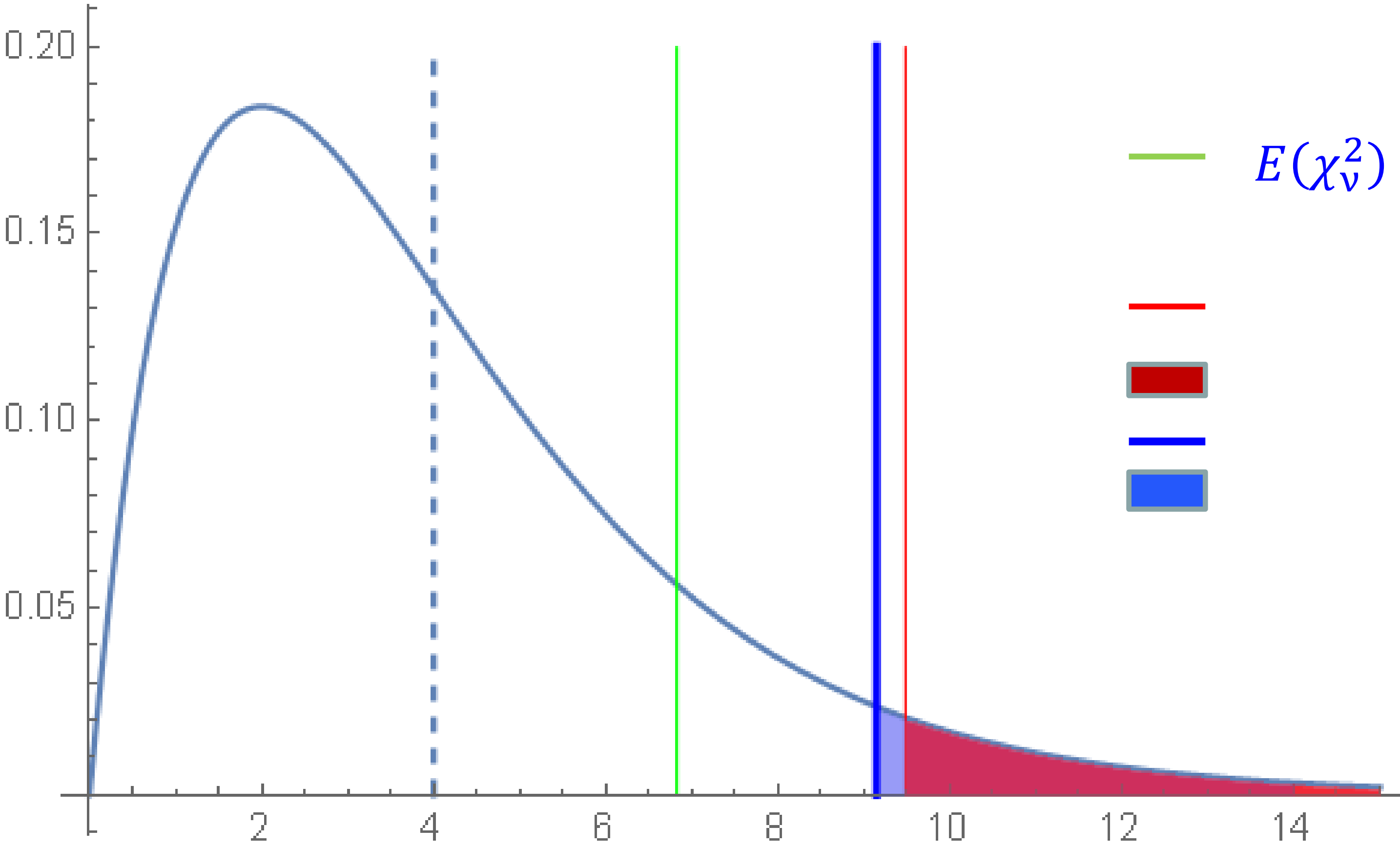
A more stringent condition is

$$\alpha = \Pr\{\chi^2_\nu > [E(\chi^2_\nu) + \sigma(\chi^2_\nu)]\} = \Pr\{\chi^2_\nu > (\nu + \sqrt{2\nu})\}$$

which means that, to accept consistency, χ^2_{obs} is requested to lie within one standard deviation to the right of the expectation (CODATA 98)

This is related to the *Birge ratio* $\sqrt{\chi^2_{\text{obs}}/\nu}$ used in the CODATA adjustments

Graphical example



$E(\chi^2_\nu) + \sigma(\chi^2_\nu) = 6.82$

$\alpha_1 = 0.32$

$Q_{.95} = 9.45$

$\alpha_2 = 0.05$

$\chi^2_{\text{obs}} = 9.15$

$p = 0.058$

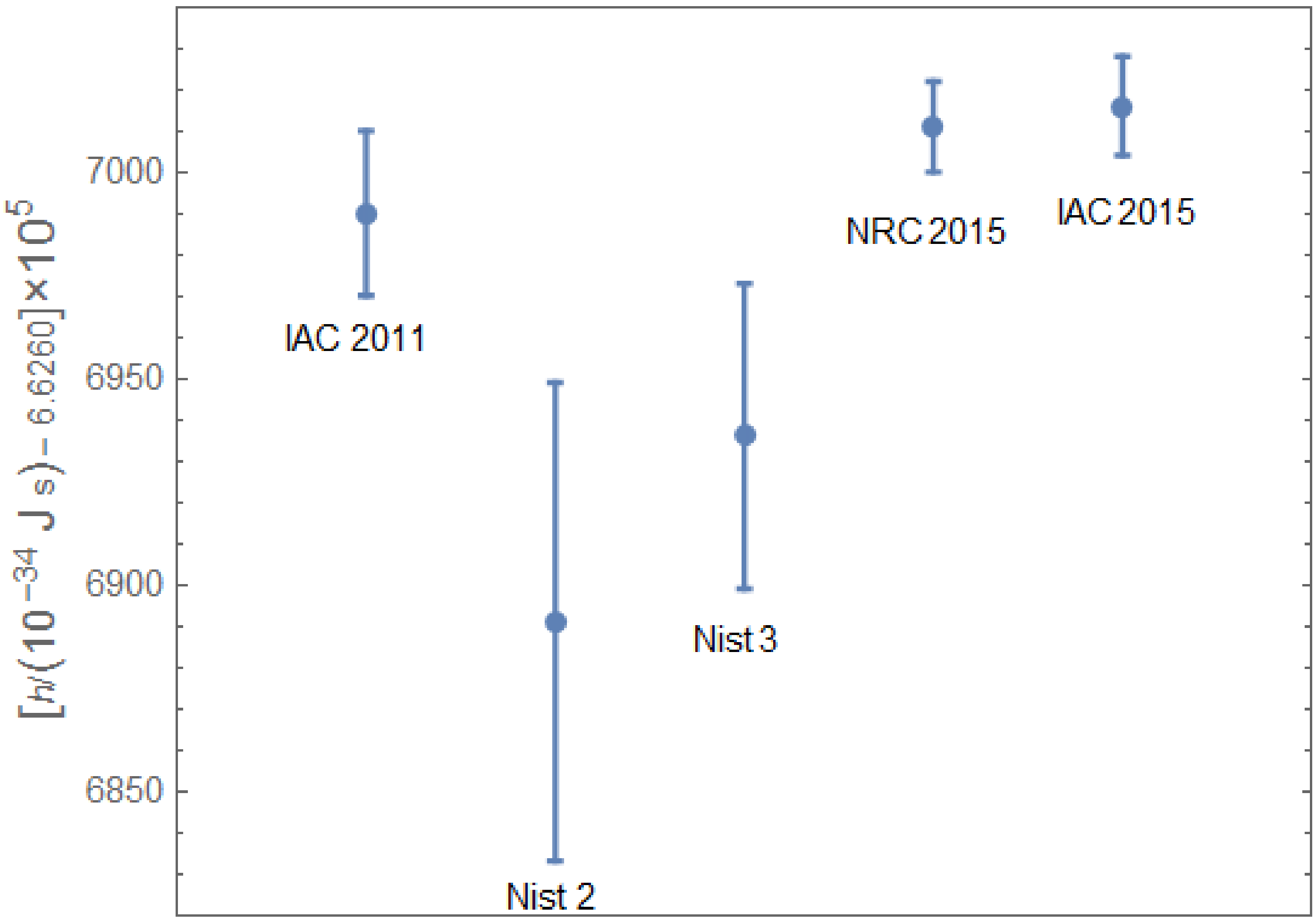
Data considered

$h = 6.626\,070\,16(13) \times 10^{-34}$ Js	2.0×10^{-8}	IAC 2015
$h = 6.626\,069\,90(20) \times 10^{-34}$ Js	3.0×10^{-8}	IAC 2011
$h = 6.626\,069\,36(37) \times 10^{-34}$ Js	5.6×10^{-8}	NIST-3
$h = 6.626\,068\,91(58) \times 10^{-34}$ Js	8.7×10^{-8}	NIST-2 (1998)
$h = 6.626\,070\,11(12) \times 10^{-34}$ Js	1.8×10^{-8}	NRC 2015
$h = 6.626\,071\,2(20) \times 10^{-34}$ Js	2.0×10^{-7}	NPL 2012

$$r(1,2) = 0.35 \qquad r(3,4) = 0.09$$

Further correlations exist due the corrections to the values of the National Prototypes, and need to be evaluated

Data considered



Considerations

1. **At least three independent experiments, including work from watt balance and XRCD experiments, yield consistent values of the Planck constant with relative standard uncertainties not larger than 5 parts in 10^8,**

The request of independence cannot be met, thus probably this request needs a broader interpretation

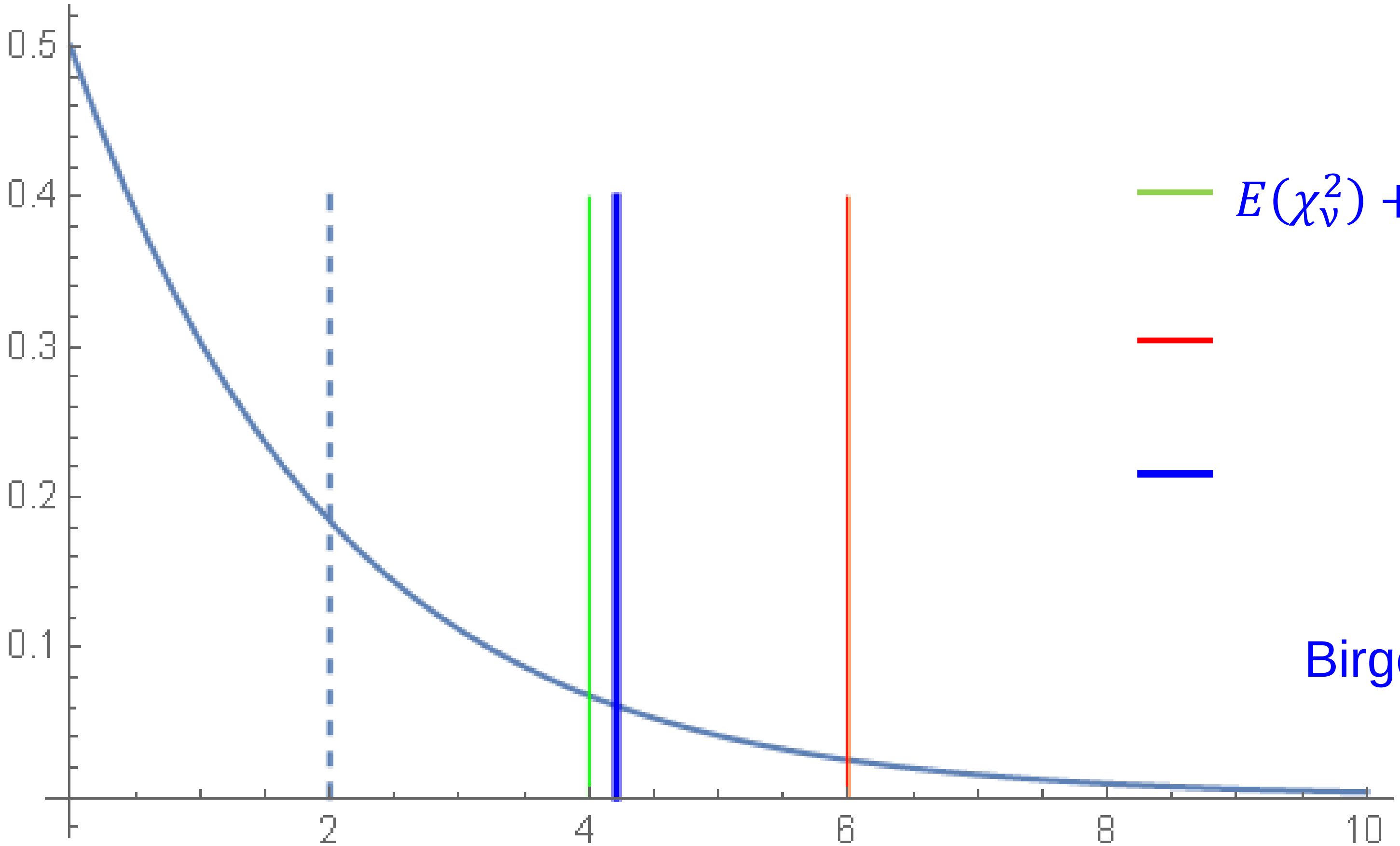
Independent of the above and of consistency considerations, the available data do not meet condition 1 in terms of relative uncertainty

Considerations

2. **At least one of these results should have a relative standard uncertainty not larger than 2 parts in 10^8,**

Condition 2 is met

Consistency
Three latest data



$E(\chi_\nu^2) + \sigma(\chi_\nu^2) = 4.0$

$\alpha_1 = 0.14$

$Q_{.95} = 5.99$

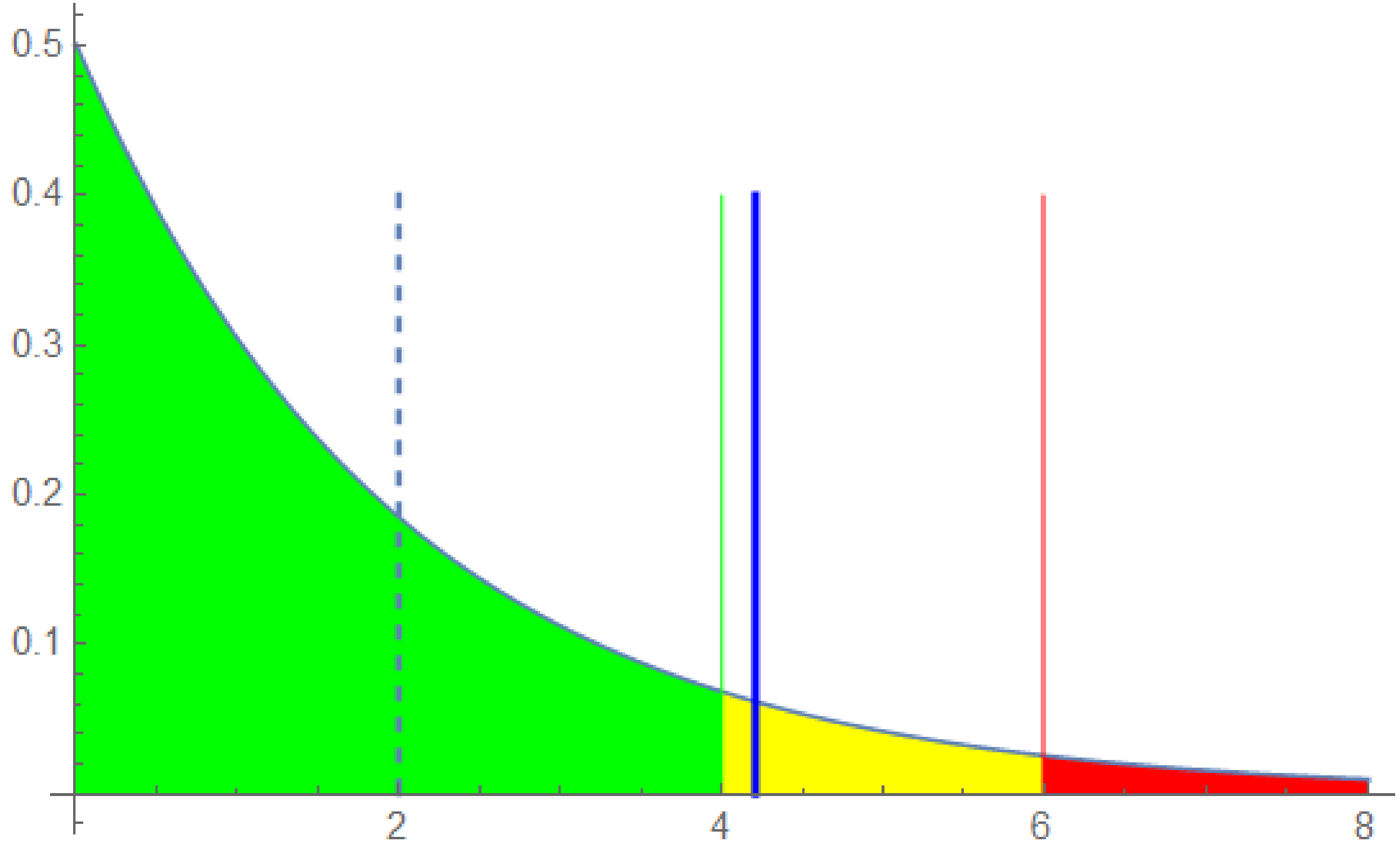
$\alpha_2 = 0.05$

$\chi^2_{\text{obs}} = 4.21$

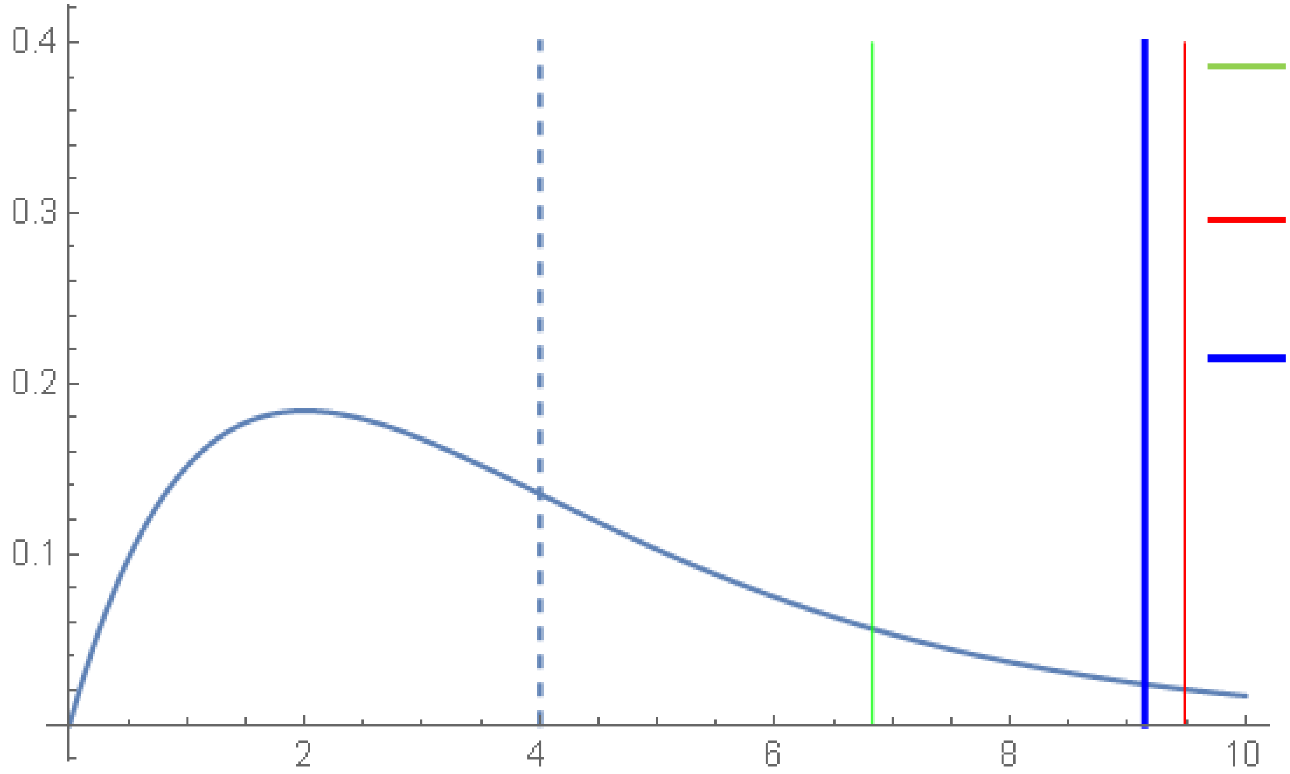
$p = 0.12$

Birge ratio= 1.45

Consistency
Three latest data



Consistency
All considered data



$E(\chi_\nu^2) + \sigma(\chi_\nu^2) = 6.83$

$\alpha_1 = 0.15$

$Q_{.95} = 9.49$

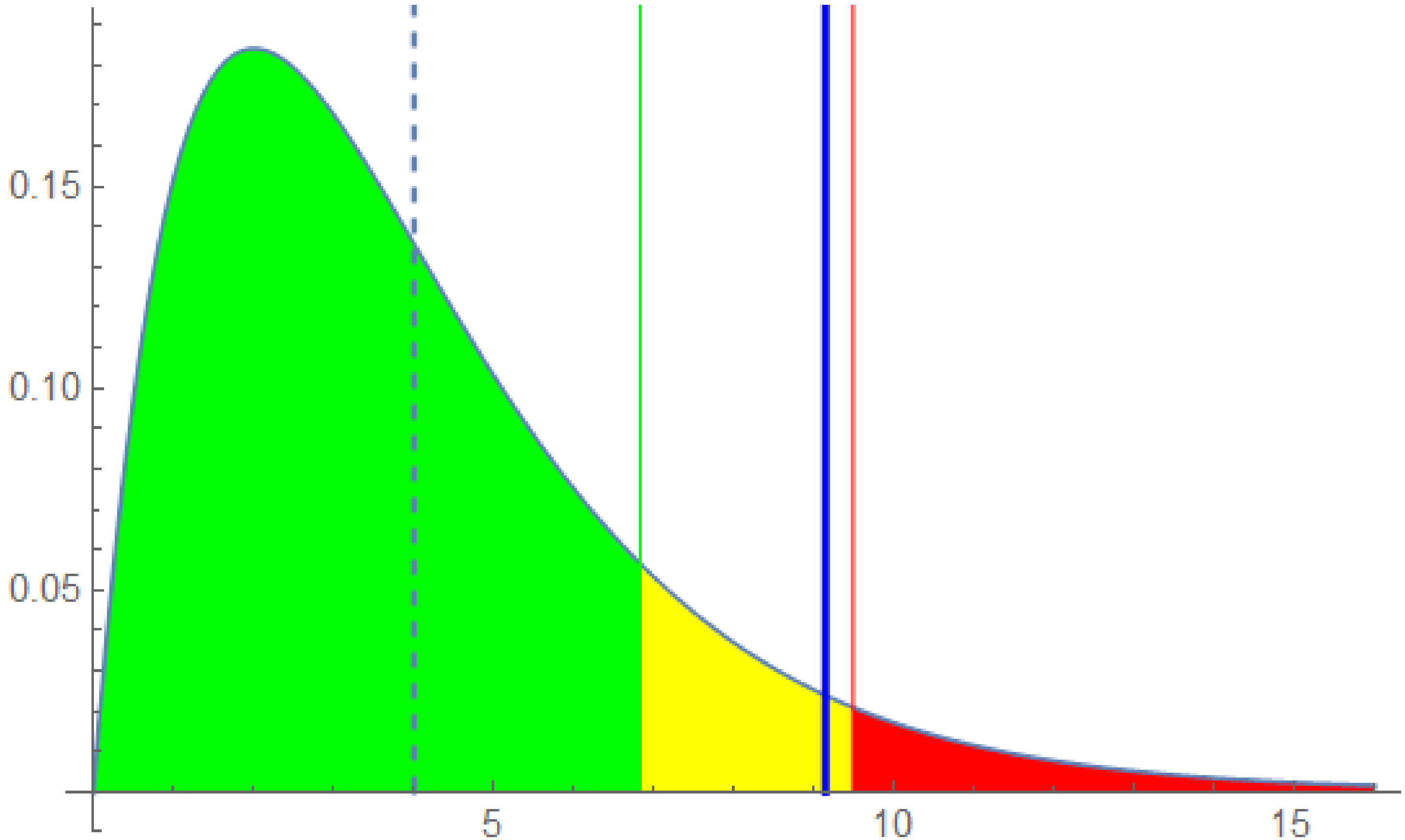
$\alpha_2 = 0.05$

$\chi^2_{\text{obs}} = 9.15$

$p = 0.058$

Birge ratio = 1.51

Consistency
All considered data



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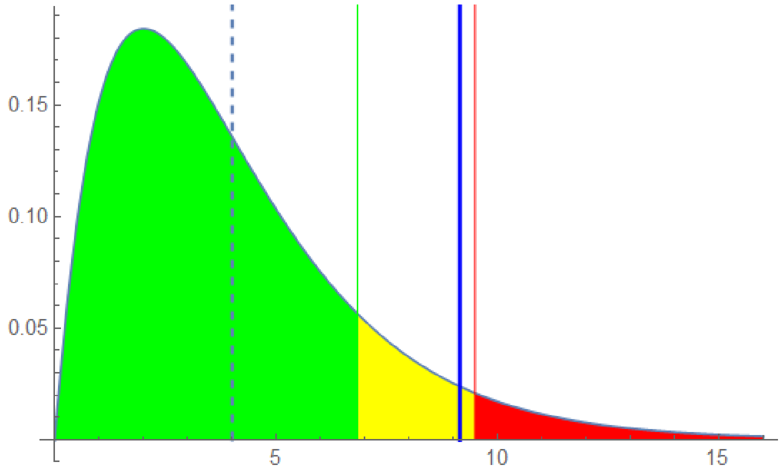
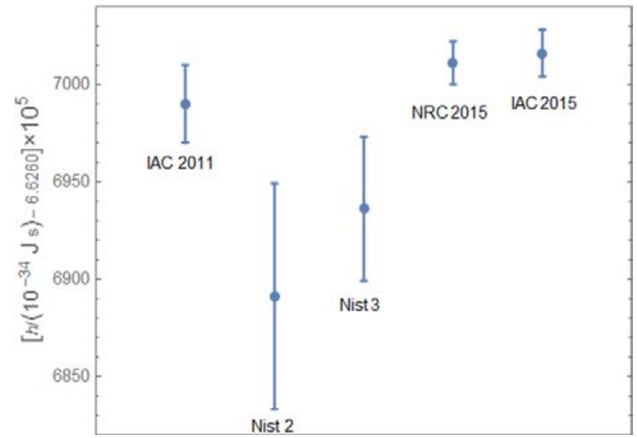
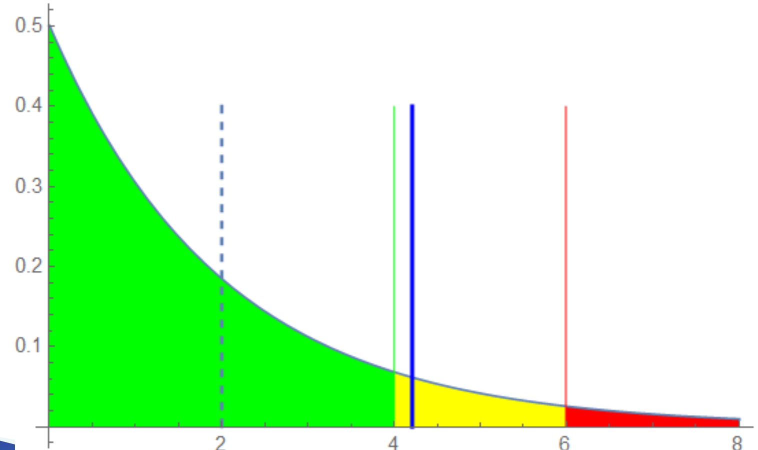
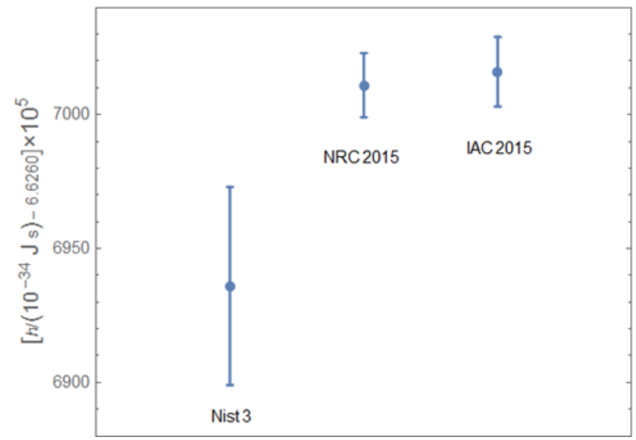
Consistency
A summary









Conclusions

- Condition 1 is not met, as regards independence and uncertainties.
- In all considered cases, data passes the test at 0.05 significance level, does not at the level corresponding to the quantile (expectation+one standard deviation).
- The statistic χ^2_{obs} is dangerously close to the 95th percentile when considering all relevant data.
- The CCM has to decide about consistency. As a personal opinion, I would be reassured by a χ^2_{obs} well within the high-density region of the PDF.
- Condition 2 is met.

Thank you for your attention