Evaluation of Uncertainty in the Adjustment of Fundamental Constants

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Motivation

- Task: combine information from multiple studies on same quantity
- Applications
  - Interlaboratory studies and key comparisons
  - Adjustment of fundamental constants
  - Meta-analysis of clinical studies
- Challenges:
  - inconsistent data
  - presence of correlations, ...

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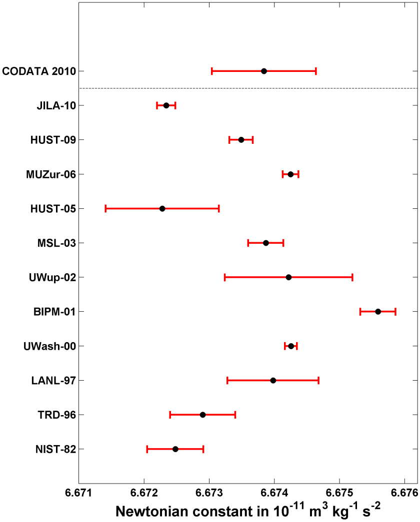
> *Combining information from disparate sources is a fundamental activity in both scientific research and policy decision making*
>
> *The process of learning is one of combining information: we are constantly called upon to update our beliefs in the light of new evidence, which may come in various forms*
>
> *How does this process work in practice?*
>
> — National Research Council (1992)
> *Combining information:*
> *Statistical issues and opportunities for research*

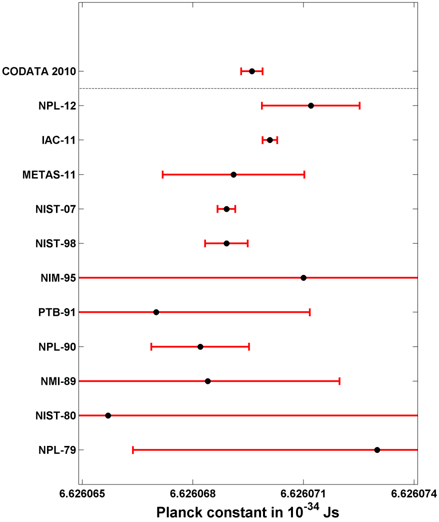
Newtonian Constant of Gravitation G



- Inconsistency of data illustrates current metrological challenges
- Ideally, resolve discrepancies by identifying metrological reasons
- Need for consensus value: statistical (adjustment) problem

P. J. Mohr, B. N. Taylor and D. B. Newell (2012) CODATA Recommended Values of the Fundamental Physical Constants: 2010 *Reviews of Modern Physics* **84**, 1527-1605.

Planck Constant h



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Problem

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- Over-dispersion suggests that there may be uncertain components that are yet unrecognized in the corresponding, laboratory-specific uncertainty budgets

Dark uncertainty — Thompson & Ellison (2011)

M. Thompson and S. L. R. Ellison (2011). Dark uncertainty. *Accreditation and Quality Assurance* **16**, 483-487.

Solutions

- Downweigh or discard measurement results that are markedly discordant with bulk of the others

M. G. Cox (2002) The evaluation of key comparison data *Metrologia* **39**, 589-595.

M. G. Cox (2007) The evaluation of key comparison data: determining the largest consistent subset *Metrologia* **44**, 187-200.

G. Ratel (2006) Median and weighted median as estimators for the key comparison reference value (KCRV) *Metrologia* **43**, S244-S248.

A. L. Rukhin and A. Possolo (2011) Laplace random effects models for interlaboratory studies *Computational Statistics and Data Analysis* **55**, 1815-1827.

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- Expand stated uncertainties by common multiplicative factor sufficiently to achieve mutual consistency (Birge Ratio)

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B. Toman and A. Possolo (2009) Laboratory effects models for interlaboratory comparisons *Accreditation and Quality Assurance* **14**, 553-563.

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- Expand stated uncertainties by common multiplicative factor sufficiently to achieve mutual consistency (Birge Ratio)
- Explicitly acknowledge presence of yet unidentified, laboratory-specific uncertainty components, and take them into account when estimating measurand and evaluating associated uncertainty
- Bayesian model averaging has been suggested for the estimation of particular laboratory effects models

C. Elster and B. Toman (2010) Analysis of key comparisons: estimating laboratories' biases by a fixed effects model using Bayesian model averaging *Metrologia* **47**, 113-119.

Conventional Method

Measurement results from n laboratories: $(x_1, u_1), \ldots, (x_n, u_n)$

- Weighted Mean

$$\boxed{\widehat{\mu}_{\mathrm{W}} = \frac{\sum_{i=1}^{n} \frac{x_i}{u_i^2}}{\sum_{i=1}^{n} \frac{1}{u_i^2}}, \quad u(\widehat{\mu}_{\mathrm{W}}) = \left(\sum_{i=1}^{n} \frac{1}{u_i^2}\right)^{-\frac{1}{2}}}$$

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- Birge Ratio $R_{\mathrm{B}} = \sqrt{\frac{\chi^2_{\mathrm{obs}}}{n-1}}$, $\chi^2_{\mathrm{obs}} = \sum_{i=1}^{n} \frac{(x_i - \widehat{\mu}_{\mathrm{W}})^2}{u_i^2}$

$$\boxed{\widehat{\mu}_{\mathrm{B}} = \widehat{\mu}_{\mathrm{W}}, \quad u(\widehat{\mu}_{\mathrm{B}}) = R_{\mathrm{B}}\, u(\widehat{\mu}_{\mathrm{W}})}$$

  - Fails to recognize uncertainty associated with R_{B}
  - Assumes that same multiplicative inflation factor should be applied to all of $u_1, \ldots, u_n$

Statistical Models

Location-Scale Model (LSM)

$$X_i = \mu + \sigma_{\mathrm{B}} \epsilon_i, \quad \text{with} \quad \epsilon_i \sim N(0, u_i^2) \quad \text{for} \quad i = 1, \ldots, n$$

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Random Effects Model (REM)

$$X_i = \mu + \lambda_i + \epsilon_i, \quad \text{with} \quad \lambda_i \sim N(0, \sigma_\lambda^2), \quad \epsilon_i \sim N(0, u_i^2) \quad \text{for} \quad i = 1, \ldots, n$$

- $\lambda_1, \ldots \lambda_n$ denote random effects independent of $\epsilon_1, \ldots, \epsilon_n$
- There may be correlations between $\lambda_1, \ldots, \lambda_n$ or between $\epsilon_1, \ldots, \epsilon_n$
- σ_{B} and σ_λ determine *Dark Uncertainty*

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LSM amplifies lab-specific uncertainties by common factor $\sigma_{\mathrm{B}} > 1$

REM introduces additive laboratory effects, with common variance σ_λ^2

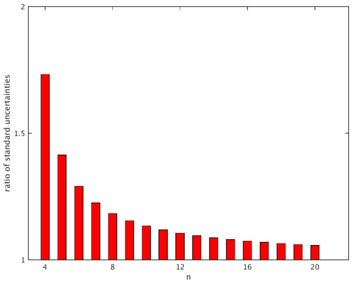
Model Fitting

Both LSM and REM may be fitted to data using anyone of several different methods for estimating μ and σ_{B} (for LSM), and μ and σ_λ (for REM)

Location-scale model

- Weighted mean for μ and method of moments for σ_{B}
  - R. T. Birge 1932 The calculation of errors by the method of the least squares. *Physical Reviev* **40**, 207-227.
- Bayesian estimates for μ and σ_{B}
  - O. Bodnar and C. Elster (2014) On the adjustment of inconsistent data using the Birge ratio. *Metrologia* **51**, 516-521.
  - B. Toman, J. Fischer and C. Elster (2012) Alternative analyses of measurements of the Planck constant. *Metrologia* **49**, 567-571.

Bayesian Improvement of Birge Ratio Procedure



- Ratio of standard uncertainties obtained by the modified Birge adjustment and the Birge adjustment as a function of the number n of adjusted values
- O. Bodnar and C. Elster (2014) On the adjustment of inconsistent data using the Birge ratio. *Metrologia* **51**, 516-521.

  B. Toman, J. Fischer and C. Elster (2012) Alternative analyses of measurements of the Planck constant. *Metrologia* **49**, 567-571.

Random effects model

- First estimate σ_λ, then estimate μ by weighted mean with weights proportional to $\{1/(\widehat{\sigma}_\lambda^2 + u_i^2)\}$

Viechtbauer (2010) Conducting meta-analyses in R with the metafor package. *Journal of Statistical Software* **36 (3)**, 1-48

A. L. Rukhin (2013) Estimating heterogeneity variance in meta-analysis. *Journal of the Royal Statistical Society Series B* **75**, 451-469.

- (Restricted) Maximum Likelihood

A. L. Rukhin and M. G. Vangel (1998) Estimation of a common mean and weighted means statistics. *Journal of the American Statistical Association* **93**, 303-308.

S. R Searle, G. Casella and C. E. McCulloch (2006) *Variance Components*. John Wiley & Sons, New Jersey.

B. Toman and A. Possolo (2009) Laboratory effects models for interlaboratory comparisons. *Accreditation and Quality Assurance* **14**, 553-563.

- Bayesian methods

A. Gelman (2006) Prior distributions for variance parameters in hierarchical models (Comment on article by Browne and Draper). *Bayesian Analysis* **1**, 515-534.

O. Bodnar, A. Link and C. Elster (2015) Objective Bayesian inference for a generalized marginal random effects model. To appear in *Bayesian Analysis*.

Model Fitting

BAYESIAN REFERENCE ANALYSIS for RANDOM EFFECTS MODEL

- The reference prior is most uninformative in a certain sense

J. Berger and J. M. Bernardo (1992a) Ordered group reference priors with application to the multinomial problem. *Biometrika* **79**, 25-37.

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- Analytic expression for the reference prior available
- Calculation of posterior distribution by simple 1-d quadrature, no MCMC required
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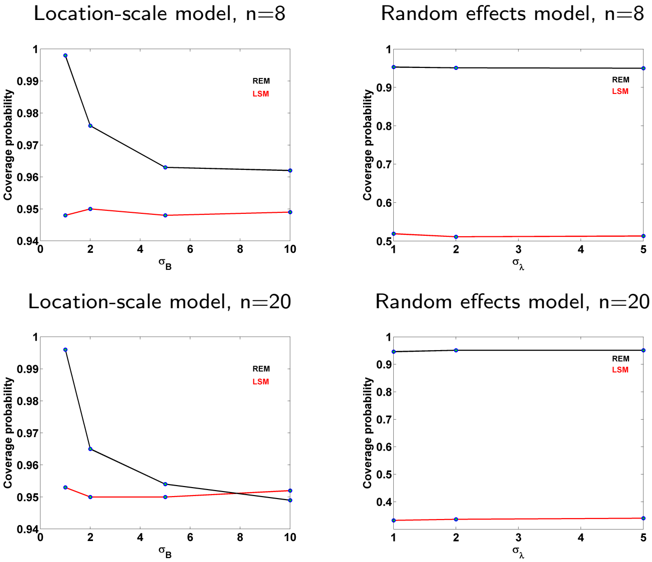
Robustness, True Model *vs.* Fitted Model



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- Accuracy of coverage probability largely maintained when REM is fitted to data consistent with LSM, but seriously degraded when LSM is fitted to data consistent with REM

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- REM achieves greater accuracy of coverage probability than LSM when the $\{\epsilon_i\}$ or the $\{\lambda_i\}$ are correlated, and when both the $\{\epsilon_i\}$ and the $\{\lambda_i\}$ have rescaled Student's t_5 distributions

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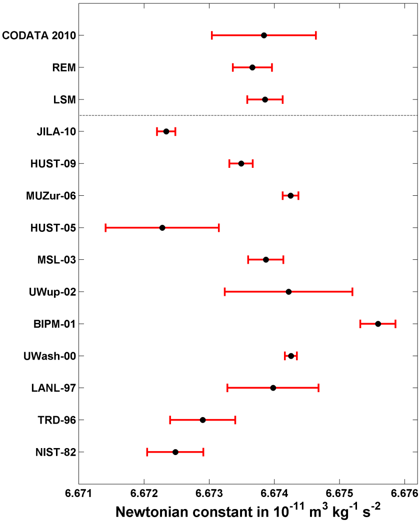
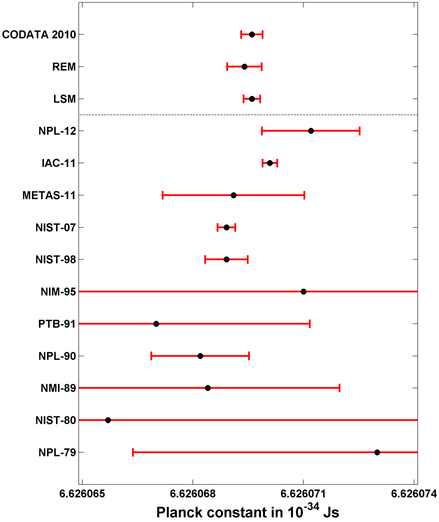


- LSM — Conventional estimate and uncertainty evaluation based on Birge Ratio
- REM — Bayesian estimate and uncertainty evaluation based on reference prior

Planck Constant h



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- Random effects model (REM) is more robust than location-scale model (LSM)
- Fitting procedure influential: Bayesian reference analysis appears to work well
- Bayesian reference analysis in conjunction with REM assigns uncertainties to consensus values for G and h that are substantially different from CODATA's

Summary & Conclusions (2/2)

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- Ideally, application of statistical techniques and interpretation of corresponding results should be done in *collaboration between metrologists and statisticians*