The GUM and its (published) Supplements

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- “In art it is hard to say anything as good as saying nothing.”
  Ludwig Wittgenstein, *Culture and Value*

- The same applies to the GUM and to uncertainty in general (and to most other subjects!)

- A search for *uncertainty* on Google yields about 8.0×10^7 results, among which 2.3×10^7 books

- *Measurement uncertainty* scores 6.2×10^7, and

- *Guide to the expression of uncertainty in measurement* yields an impressive 2.2×10^7

GUM – JCGM 100:2008 and 100:201X

- First publication in 1993
- Reprint in 1995 with some corrections
- JCGM 100:2008 (free of charge) GUM 1995 with minor modifications
- Until now, a large number of documents based on the GUM has been written. The GUM has been translated into many languages
- In addition, the GUM has been adopted as a standard, in some cases as a law, in many countries

Merits of the GUM

- Provides widely accepted guidance on measurement uncertainty
- Treats in a common way systematic and random contributions
- Rests on principles of probability and statistics
- It is accused of being difficult (by some) or simplistic (by others), which means that it is a good compromise

This *Guide* establishes general rules for evaluating and expressing uncertainty in measurement that can be followed at various levels of accuracy and in many fields — from the shop floor to fundamental research.

(GUM, 1.1 Scope)

uncertainty (of measurement)
parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand (GUM, 2.2.3, Definitions)

standard uncertainty
uncertainty of the result of a measurement expressed as a standard deviation (GUM, 2.3.1, Definitions)

To be compared with →

Uncertainty of measurement
(from JCGM 100:201X)

measurement uncertainty
uncertainty of measurement
uncertainty
parameter characterizing the dispersion of the values being attributed to a quantity, based on the information used

standard measurement uncertainty
standard uncertainty
standard deviation of the random variable describing the state of knowledge about a quantity

Uncertainty (Type A and Type B) a Type A standard uncertainty is obtained from a **probability density function** derived from an **observed frequency distribution**, while a Type B standard uncertainty is obtained from an assumed probability density function based on the degree of belief that an event will occur [often called subjective **probability**]. Both approaches employ recognized interpretations of probability.

(GUM, 3.3.5, Basic concepts)

Practical considerations Although this *Guide* provides a framework for assessing uncertainty, it cannot substitute for critical thinking, intellectual honesty and professional skill.

(GUM, 3.4.8, Basic concepts)

Evaluating standard uncertainty & Determining (combined) standard uncertainty

Caution!

The same uppercase symbol is used with two meanings:

A (physical, chemical…) quantity

The associated random variable used to describe the experimental outcome

Estimates (i.e., measured values), are denoted by the corresponding lowercase symbol

Modelling the measurement

A *measurement model* is requested, relating the *measurand* Y to the input quantities X_i

$$Y = f(X_1, X_2, \dots, X_N)$$

Often the most difficult task

(Some) guidance is given on modelling the measurement

Determining input estimates, uncertainties and covariances

One seeks the following items of information $(i = 1, \dots N)$:

N **estimates** x_i for the input quantities X_i

N **standard uncertainties** $u(x_i)$

$N(N-1)/2$ **covariances** $u(x_i, x_j)$ between pairs of input estimates

Guidance is given on estimating the input quantities and on evaluating the associated standard uncertainties (Types A and B).

Some guidance is also given on the evaluation (and removal) of input covariances.

Propagating estimates and uncertainties

The **estimate** y of the measurand Y is obtained by «propagating» the input estimates x_i through the measurement model

$$y = f(x_1, x_2, \ldots, x_N)$$

The **standard uncertainty** $u(y)$ about the measurand is obtained by propagating the input uncertainties $u(x_i)$ and covariances $u\left(x_i, x_j\right)$ using the **law of propagation of uncertainty (Gauss!!)**

$$u^2(y) = \sum_{i=1}^{N} c_i^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_i c_j u\left(x_i, x_j\right),$$

where $c_i = \frac{\partial f}{\partial x_i}$ are the sensitivity coefficients.

Comments

Significant non-linearities in the input quantities are dealt with using second-order terms (for Gaussian distributions only)

Asymmetric distributions are also addressed (cosine error)

The method provides a reliable standard uncertainty about the measurand in the vast majority, **not in the totality** of cases

Expanded uncertainty

The GUM is usually fit for the purpose of providing a standard uncertainty $u(y)$

> …it is often necessary to give a measure of uncertainty that defines an interval about the *measurement result* that may be expected to encompass a *large* fraction of the distribution of values that could reasonably be attributed to the measurand. (GUM, 6.1.2)

Expanded uncertainty $U(y) = ku(y)$

Whenever practicable, the level of confidence *p* associated with the interval defined by *U* should be estimated and stated. It should be recognized that multiplying $u(y)$ by a constant provides no new information but presents the previously available information in a different form. (GUM, 6.2.3)

Main drawbacks

- Poor guidance on the construction of a coverage interval (emphasis is on standard uncertainty), limited to a situation optimistically considered as frequently occurring, in which case a coverage probability can be assigned to expanded uncertainty

- No guidance on the (frequent) case of many measurands

Are the cases not covered in the GUM of practical importance?

- The CIPM MRA asks for CMCs at the 95 % coverage probability

- A significant number of quantities of practical importance are such that the current practice $U = ku$ (with typically $k = 2$) is inappropriate

- Any calibration of a set of artefacts, be they weights, capacitors, gauge blocks or similar, is a multivariate case

Remedies

- Coverage interval (and more): JCGM 101:2008, Supplement 1 to the GUM - Propagation of distributions using a Monte Carlo method

- Multivariate case: JCGM 102:2011, Supplement 2 to the GUM - Extension to any number of output quantities

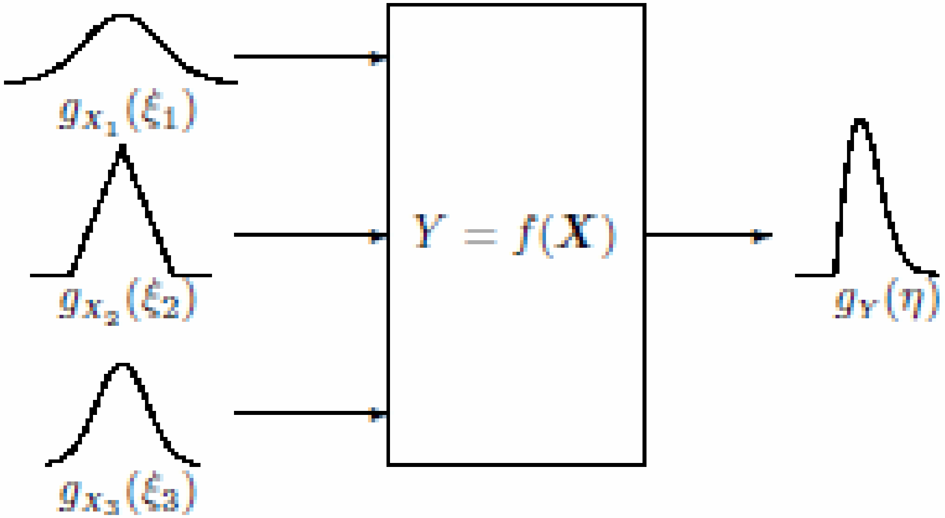
Supplement 1

The GUM is based on propagation of estimates and uncertainties, *i.e.*, first and second moments of the random variables X_i, to provide the corresponding moments of the random variable Y.

This procedure is distribution-free, whereas realistic coverage intervals depend on the probability density function (PDF) for Y.

In Supplement 1, the PDF for Y is obtained by propagating the PDFs for the X_i through the measurement model.

Supplement 1



Supplement 1

The (numerical) mechanism is the Monte Carlo method:

From each input PDF draw at random a value x_i for the random variable X_i.

Use the resulting set of values to evaluate the model, thus obtaining a corresponding value of Y. The latter is a possible value for the measurand.

Iterate M times the preceding two steps, to obtain M values for Y.

Supplement 1

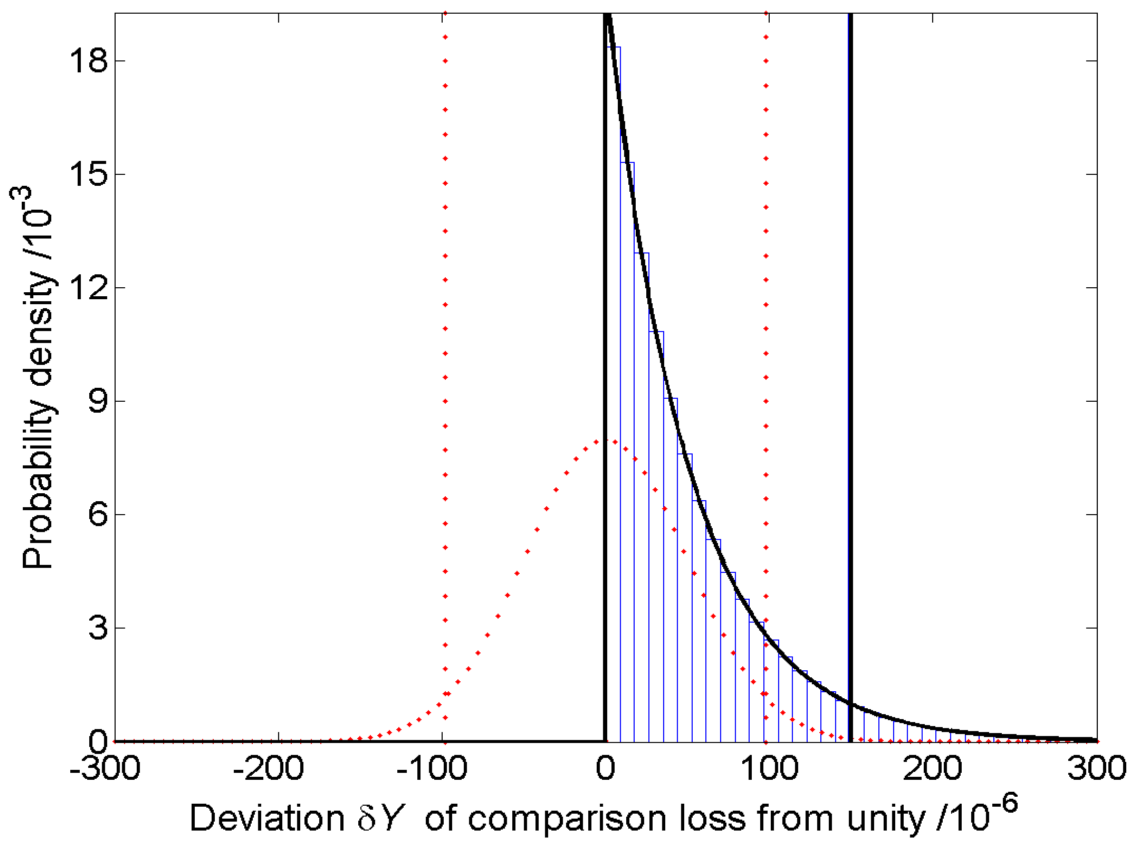
Extensive guidance given on:

assigning PDFs based on available knowledge, and

sampling at random from these PDFs

obtaining a coverage interval from the PDF for Y.

Example
(from Supplement 1)



Red dotted: GUM

Solid blue: S1

Supplement 2

Generalizes to the **multivariate** case:

The law of propagation of uncertainty, and

the construction of a (joint) PDF for the vector measurand $\boldsymbol{Y}$, from which a coverage region for $\boldsymbol{Y}$ can be obtained.

Generalized LPU

Generalization of the scalar law of propagation of uncertainty to the **multivariate** cases

explicit

$$\boldsymbol{Y}_{(m\times 1)} = \boldsymbol{f}\left(\boldsymbol{X}_{(N\times 1)}\right)$$

and implicit

$$\boldsymbol{h}(\boldsymbol{Y},\boldsymbol{X}) = \boldsymbol{0}$$

where the quantities involved can be real or complex.

Generalized LPU

By introducing **covariance matrices** associated with quantity estimates

$$\boldsymbol{U_z} = \begin{bmatrix} u(z_1, z_1) & \cdots & u(z_1, z_q) \\ \vdots & \ddots & \vdots \\ u(z_q, z_1) & \cdots & u(z_q, z_q) \end{bmatrix},$$

the generalized LPUs are

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Generalized LPU

for the explicit case: $\boldsymbol{U_y} = \boldsymbol{C_x} \boldsymbol{U_x} \boldsymbol{C_x}^{\mathrm{T}},$

where $\boldsymbol{U_x}$ and $\boldsymbol{U_y}$ are the covariance matrices of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, and

$$\boldsymbol{C_x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_N} \end{bmatrix}$$

is the **sensitivity matrix**.

Generalized LPU

For the implicit case:

$$\boldsymbol{C}_{\boldsymbol{y}}\boldsymbol{U}_{\boldsymbol{y}}\boldsymbol{C}_{\boldsymbol{y}}^{\mathrm{T}} = \boldsymbol{C}_{\boldsymbol{x}}\boldsymbol{U}_{\boldsymbol{x}}\boldsymbol{C}_{\boldsymbol{x}}^{\mathrm{T}},$$

where

$$\boldsymbol{C}_{\boldsymbol{x}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_N} \end{bmatrix}, \boldsymbol{C}_{\boldsymbol{y}} = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial y_1} & \cdots & \frac{\partial h_m}{\partial y_m} \end{bmatrix}$$

Guidance is given on how to obtain $\boldsymbol{U}_{\boldsymbol{y}}$ in a numerically efficient and stable manner.

Generalized Monte Carlo

The procedures of Supplement 1 are generalized so as to obtain a numerical approximation of the (joint) PDF for the vector measurand $\boldsymbol{Y}$, from which a **coverage region** for $\boldsymbol{Y}$ can be obtained.

Three regions are considered: hyper-ellipsoidal, hyper-rectangular (typically conservative) and (approximately) smallest volume (with no regular shape).

GUM *vs* Supplements

- Uncertainties (and estimates) are:
  - estimates of moments of frequency distributions, in the current GUM (they have degrees of freedom)
  - exact moments of state-of-knowledge distributions, in the Supplements (no degrees of freedom)

A considerable inconsistency

- La incertidumbre, margarita cuyos pétalos no se termina jamás de deshojar, ...
- Uncertainty, a daisy whose petals are pulled off forever

(La Tia Julia y el Escribidor, Aunt Julia and the Scriptwriter, 1977, Mario Vargas Llosa)

Uncertainty is a personal matter; it is not *the* uncertainty but *your* uncertainty.

Dennis Lindley, *Understanding Uncertainty* (2006)

Thank you for your attention