Case study: Bayesian analysis of a flow meter calibration problem

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G J P Kok, A M H van der Veen, P M Harris, I M Smith and C. Elster (2015)
Bayesian analysis of a flow meter calibration problem, ***Metrologia, 52, 400-405***

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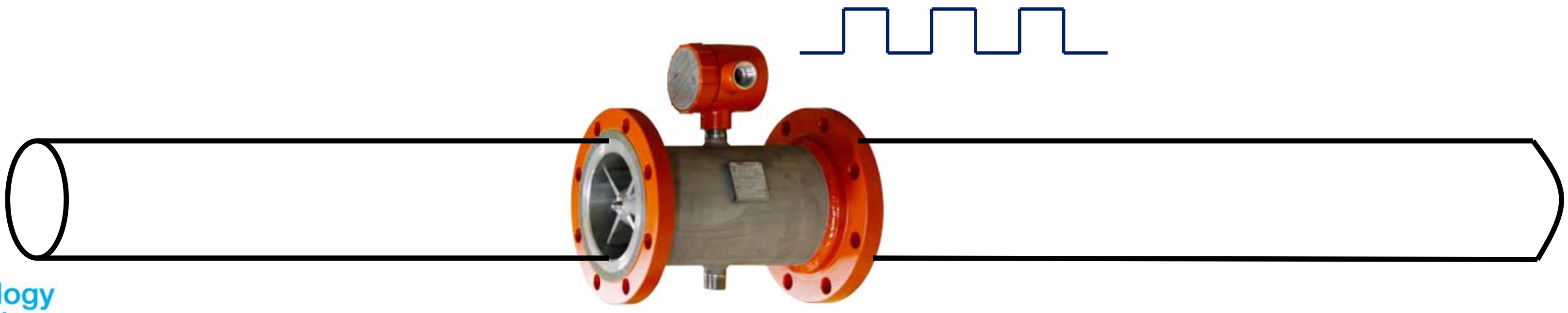
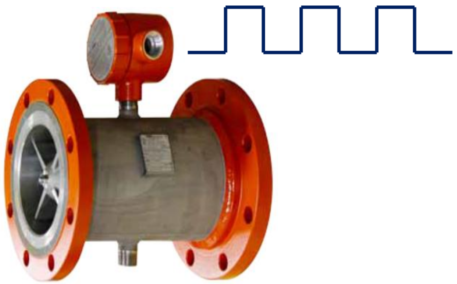
Outline

- Bayesian analysis of flow meter calibration problem
  - An example of a normal linear regression problem
- Treatment of different prior knowledge
  - About flow meter calibration curve
  - About repeatability of flow meter
- Comparison with classical ordinary least squares (OLS) approach



Background



- Flow meter measures the volume of fluid flowing through the meter per unit of time, e.g. reported in L min^{-1}
- High-quality meter will have an electrical pulse output such that each pulse corresponds to a fixed volume flowing through the meter
- Frequency of the pulse output is proportional to flow rate q, and the proportionality constant is the K-factor k
- A calibration involves determining the K-factor for several known flow rates, and fitting a curve to the calibration data
- That calibration curve is used to provide an estimate of the K-factor at any flow rate within the measurement range of the device

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Prior knowledge regarding flow meter calibration

- Specified K-factor by manufacturer (constant)
  -> correct within 1 %

- Calibration curve of previous calibration
  -> correct within 0.1 %

- Typical variation in curve
  -> maximum and minimum values differ by less than 0.2 %

- Information on repeatability specified by manufacturer
  -> estimate of 0.025 % with degree of belief ν_0

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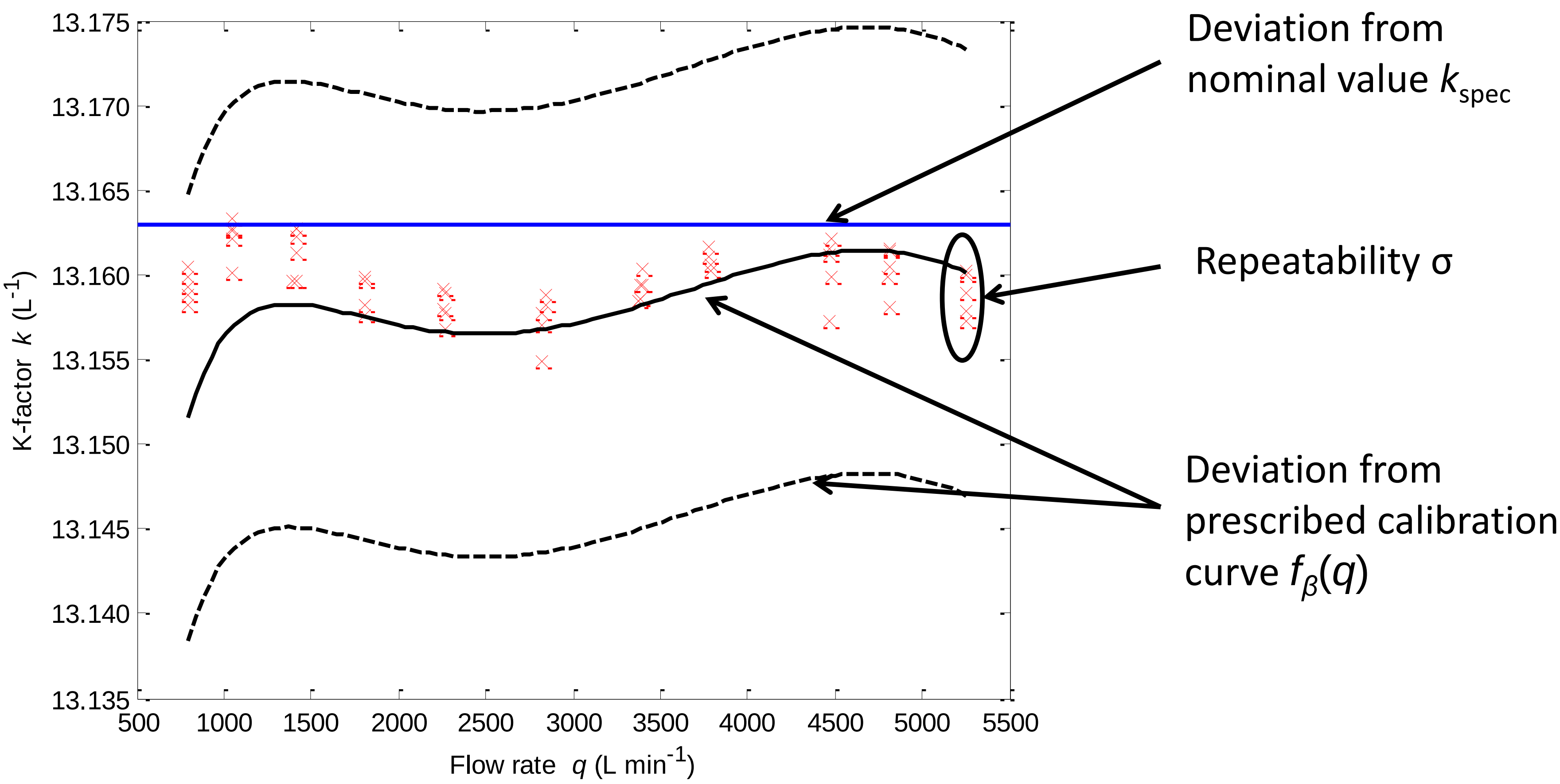
Data and some of the prior knowledge



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Statistical model

- Calibration data

$$(q_i, k_i), \quad i = 1, \ldots, n$$

- Calibration curve

$$f_\beta(q) = \beta_1 + \beta_2 / q + \beta_3 q + \beta_4 q^2 + \beta_5 q^3$$

- Statistical model

$$k_i = f_\beta(q_i) + \varepsilon_i, \qquad \varepsilon_i \sim \mathrm{N}(0, \sigma^2)$$

- Normal linear regression problem with parameters $\boldsymbol{\beta}$ and σ^2

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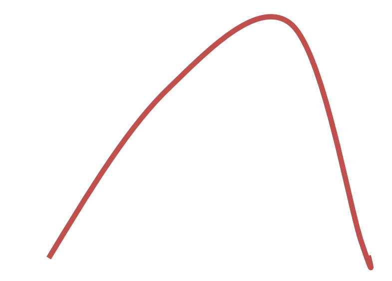
Bayesian analysis

Prior knowledge

Data &
statistical model

Posterior knowledge



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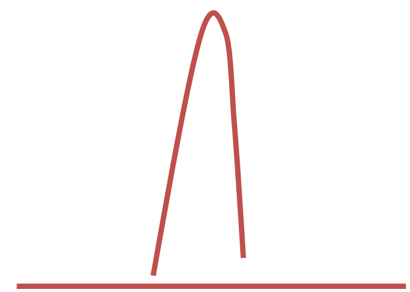
Likelihood function

Bayes theorem

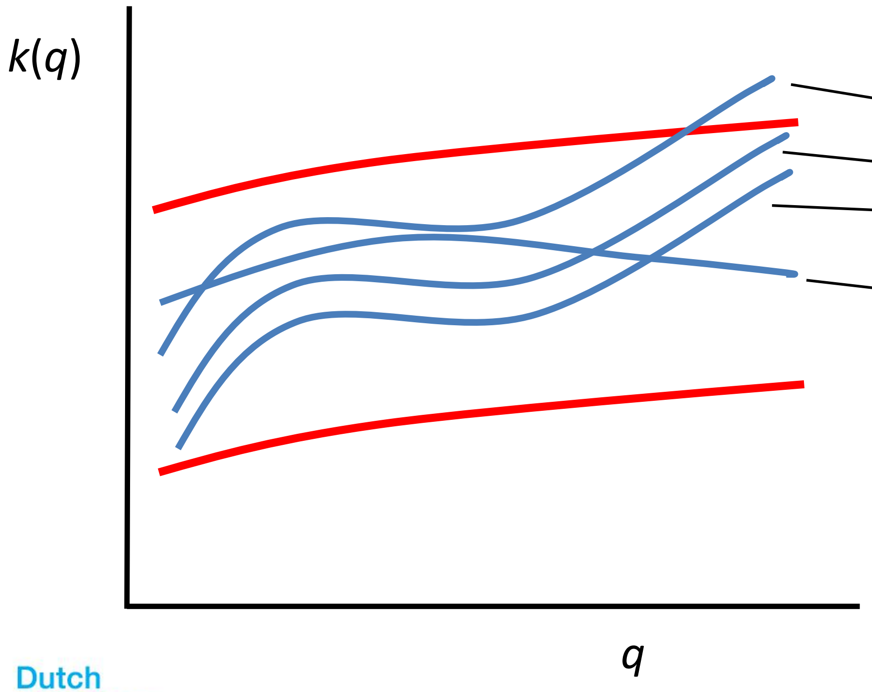


Prior distribution

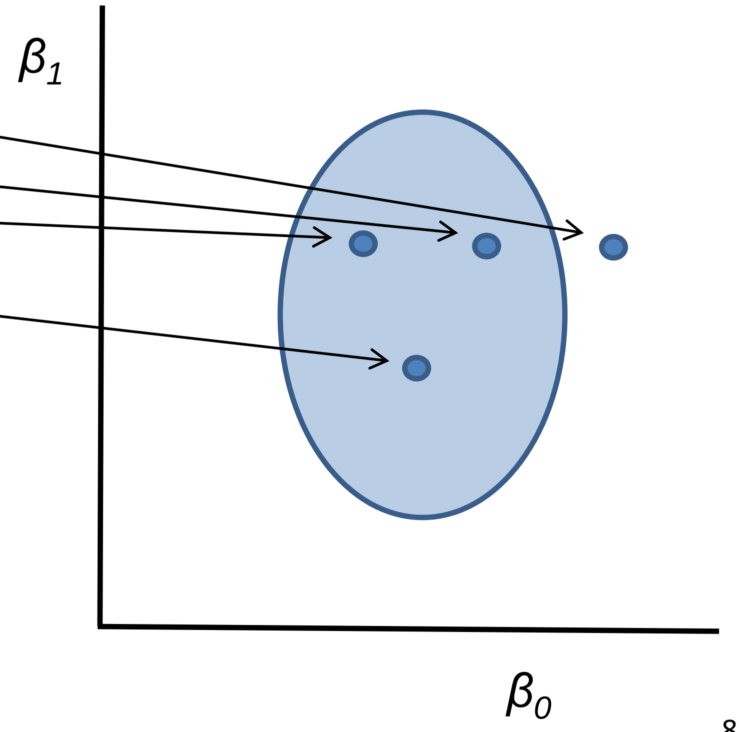
Posterior distribution

Prior distribution for curve parameters $\boldsymbol{\beta}$

Feasible curves



Feasible values of $\boldsymbol{\beta}$
(2D-cut of 5D-space)



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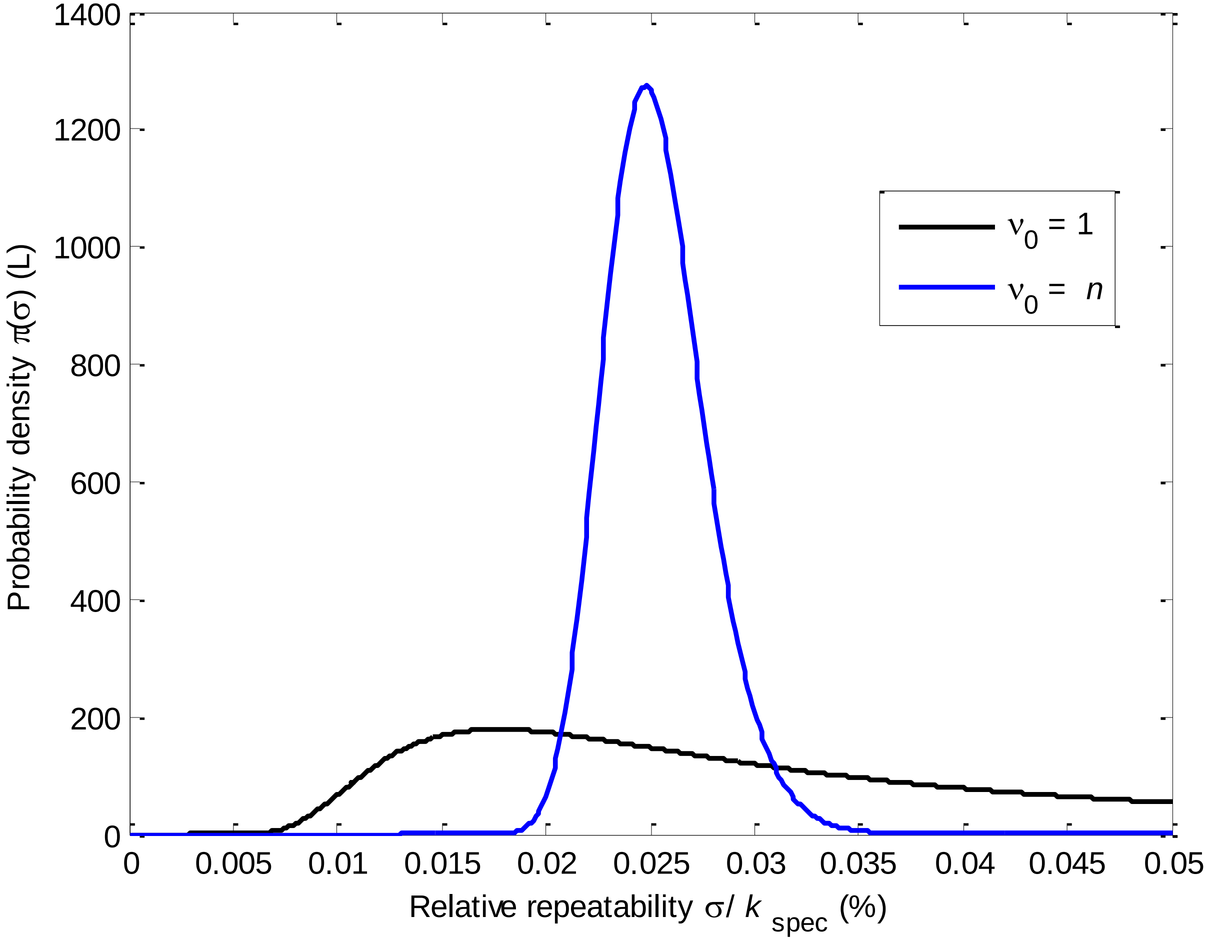
Prior distribution for repeatability σ

Inverse-Gamma($\nu_0 / 2, \nu_0 \sigma_0^2 /2$) with $\sigma_0 = 0.025$ %



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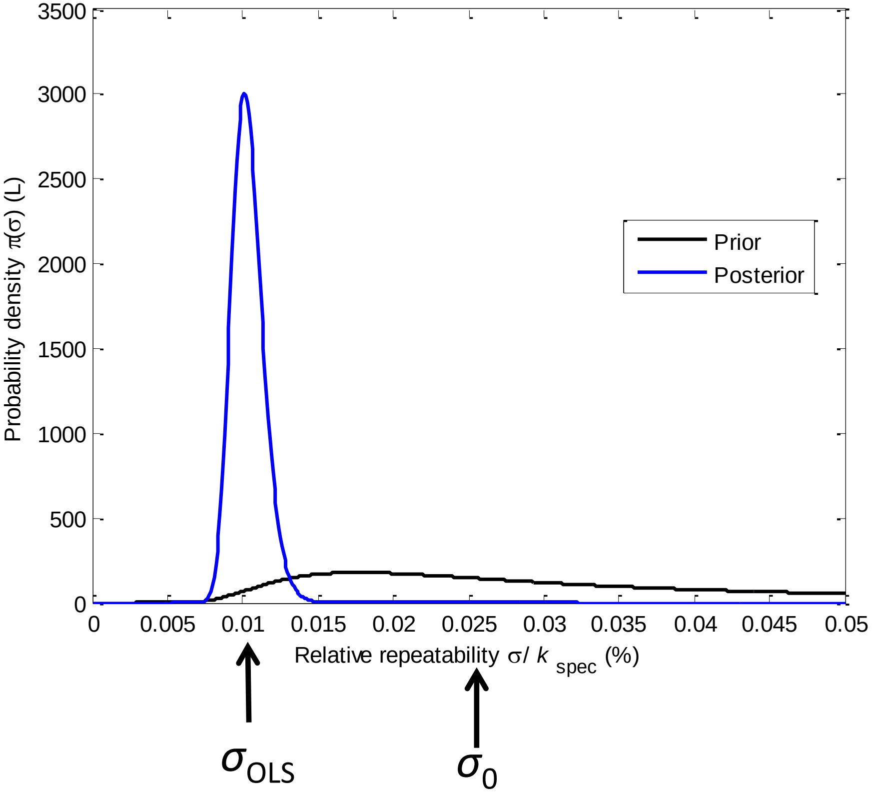
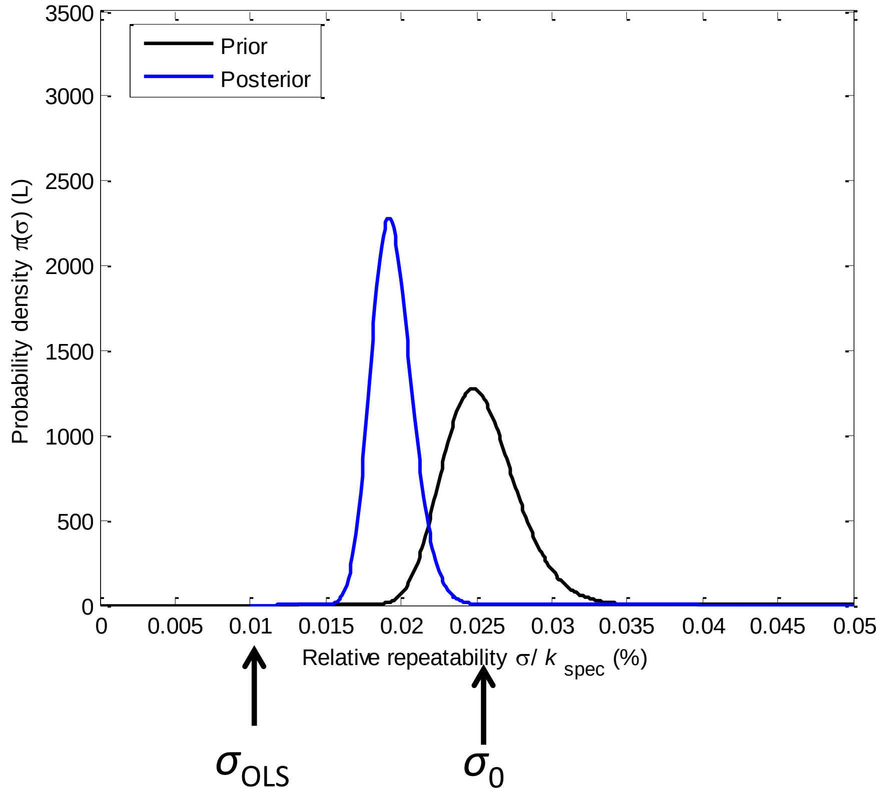
Prior and posterior distributions for σ

$v_0 = 1$



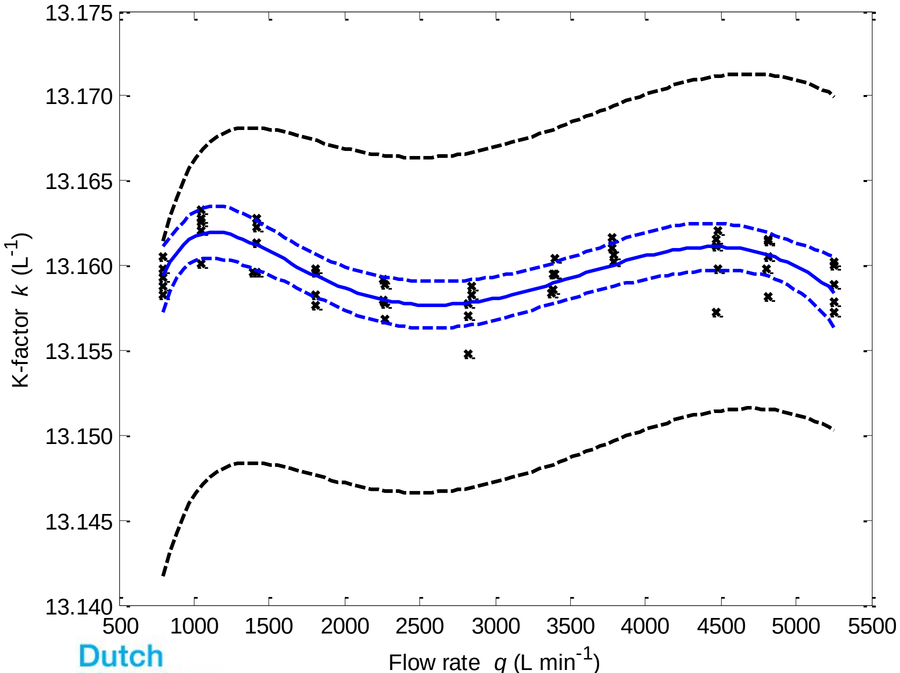
$v_0 = n$

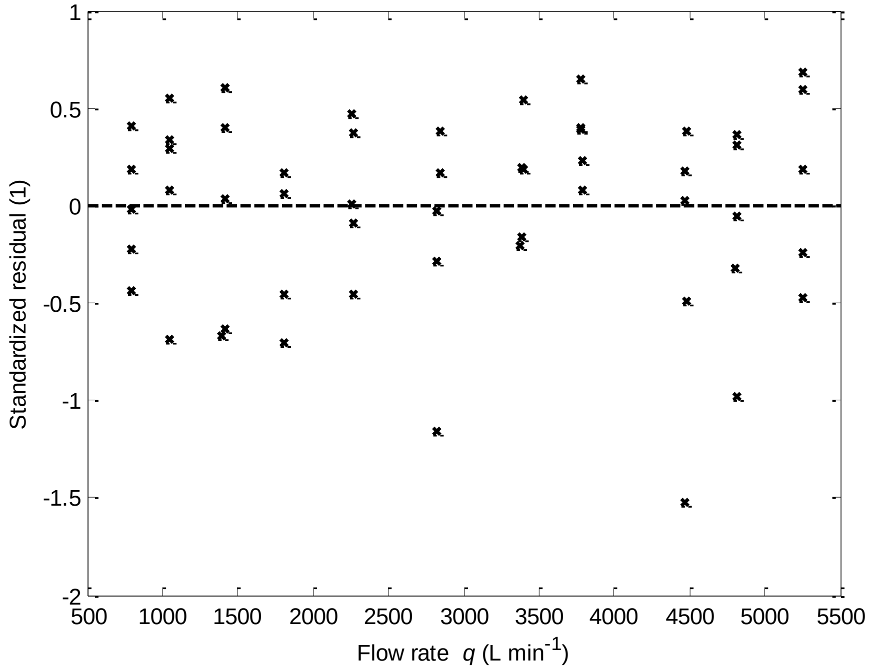


- As v_0 -> 0, the data dominates (as represented by the OLS estimate σ_{OLS})
- As v_0 -> ∞, the prior dominates (as represented by the prior estimate σ_0)

Estimate of the calibration curve

Calibration curve $f_{\beta^\wedge}(q)$, point-wise 95 % credible intervals, and constraints on calibration curve





Standardized residuals $(k_i - f_{\beta^\wedge}(q_i)) / \sigma^\wedge$

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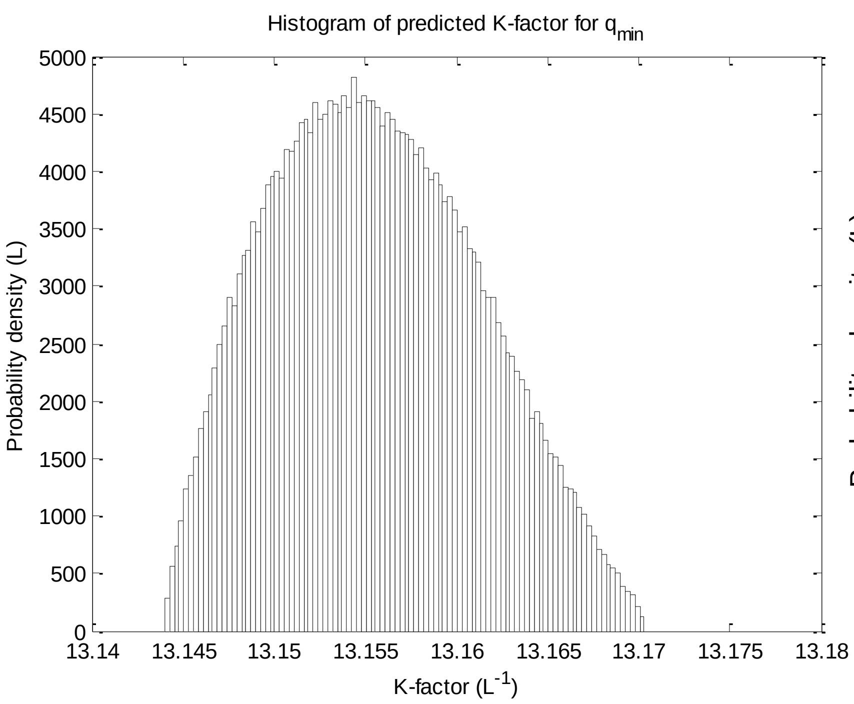
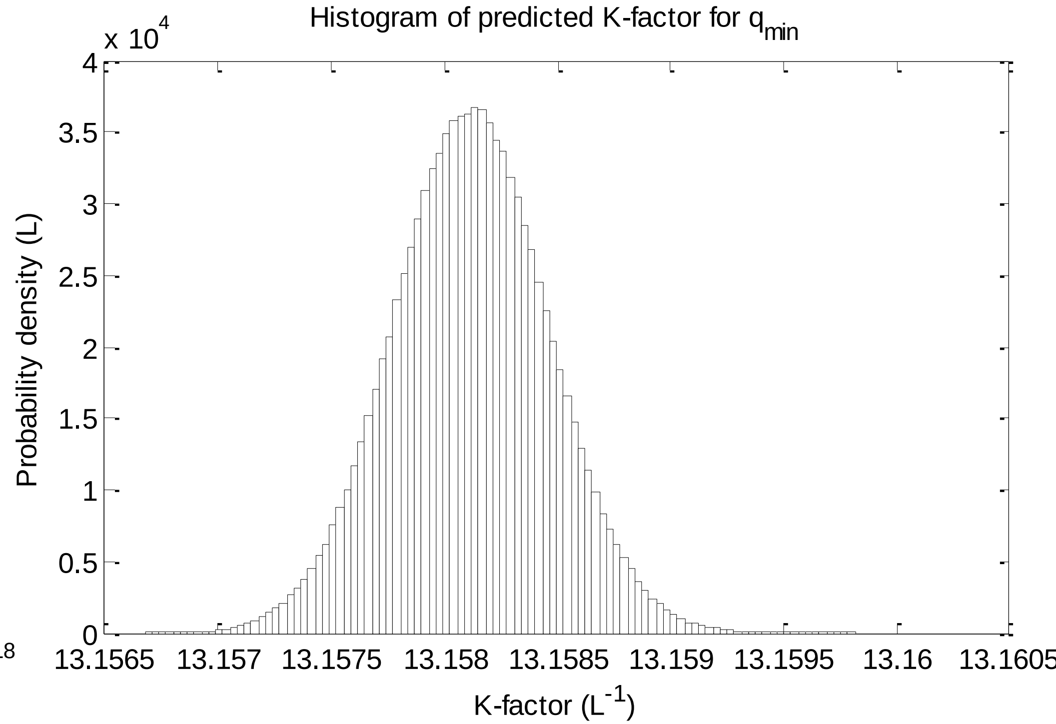
Prior and posterior distribution for K-factor for particular flow rate





Prior distribution

(appr.:13.144 to 13.171)

Posterior distribution

(appr.: 13.157 to 13.159)

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Numerical results and comparison

Approach	$k_\beta(q_{\mathrm{min}})$ (L^{-1})	$k_\beta(q_{\mathrm{mid}})$ (L^{-1})	$k_\beta(q_{\mathrm{max}})$ (L^{-1})
OLS (σ unknown)	13.159 47 (56)	13.158 12 (33)	13.158 41 (54)
BAY ($\nu_0 = 1$)	13.159 47 (59)	13.158 12 (35)	13.158 41 (57)
BAY ($\nu_0 = m$)	13.159 47 (111)	13.158 12 (65)	13.158 41 (107)
OLS ($\sigma = \sigma_0$)	13.159 47 (142)	13.158 12 (83)	13.158 41 (137)

- Estimates of K-factors are equal for all statistical approaches
- Standard uncertainties of K-factors vary depending on the weight given to the prior information on the repeatability

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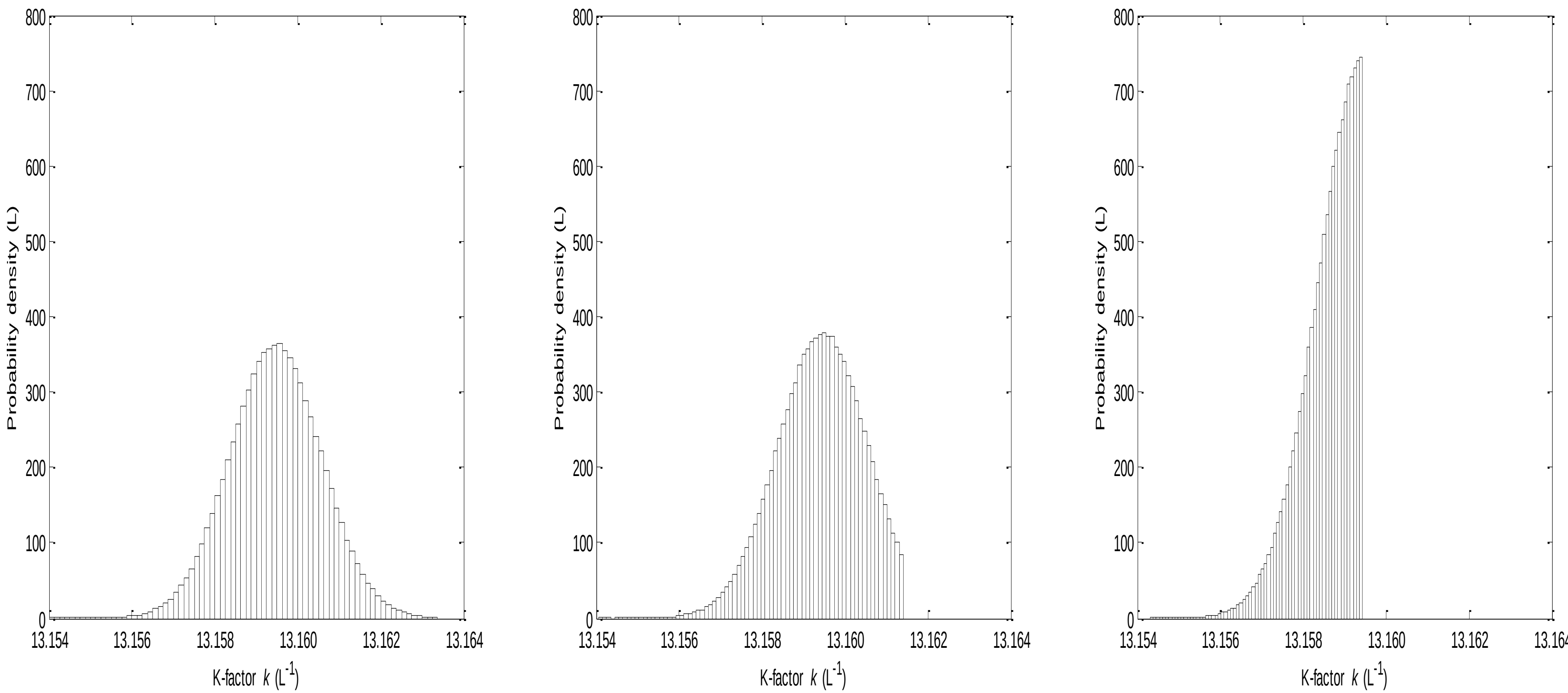
Influence of prior information

Distributions for $f_\beta(q_{\min})$ as constraint relating to previous calibration becomes tighter



Numerical results and comparison

(case: more informative prior knowledge on calibration curve)

Approach	$k_\beta(q_{\mathrm{min}})$ (L^{-1})	$k_\beta(q_{\mathrm{mid}})$ (L^{-1})	$k_\beta(q_{\mathrm{max}})$ (L^{-1})
OLS (σ unknown)	13.159 47 (56)	13.158 12 (33)	13.158 41 (54)
BAY ($\nu_0 = 1$)	13.158 97 (35)	13.158 08 (35)	13.158 39 (57)

- Estimates of K-factors are different, especially for smallest flow rate
- Standard uncertainties of K-factor at smallest flow rate becomes smaller, as prior knowledge ‘actively adds information’ in this case
- Standard uncertainties of K-factors at other flow rates are almost identical to the values before

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Final remarks

- Numerical algorithm is quite straightforward and only uses methods similar to GUM Supplement 1.

- Treatment does not answer questions about whether the flow meter ‘conforms to specifications’ about its repeatability and calibration curve
  - By interpreting those ‘specifications’ as prior knowledge, the flow meter is forced to conform!

- Use of ‘strict’ constraints leads to posterior distributions that are truncated at the boundaries of the constraints
  - Alternative forms of prior distributions, which associate small probabilities with ‘infeasible’ calibration curves, might be considered to be more realistic but are more difficult to cope with numerically.

Conclusion and acknowledgment

- Bayesian analysis permits both prior information about the flow meter and observed data to be used.

- In contrast, Ordinary Least Squares only treats the extreme cases where any of the unknowns (regression parameters $\boldsymbol{\beta}$ and variance of the data σ^2) is either completely known or unknown.

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