Bayesian uncertainty analysis for repeated measurements affected by a systematic error and application to conformity assessment

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Topic

- Bayesian analysis applied to conformity assessment of mass-produced products
- Incorporating measurement uncertainty (MU) into the conformity assessment criterion

Another Bayesian analysis: Why?

- Need to mix two fundamentally different types of information:
  - Sampling distribution of the mass-produced products
  - State-of-knowledge PDF of the measurement error

Motivation

- Electromagnetic Compatibility (EMC) compliance criterion of mass-produced products was recently under debate*
- This triggered a reconsideration of the compliance criterion

* The debate did not concern incorporation of MU into the compliance criterion

The CISPR* 80 % / 80 % rule

- *At least 80 % of the products shall comply with the emission limit with a probability of not less than 80 %*
- The 80 % / 80 % rule was introduced via CISPR Recommendation 46/1 and 46/2 in 1961
- Adopted by generic EMC emission standards and several product standards
- Used by market surveillance authorities
- Protects the consumer from appliances with a too high radio interference level
- Similar statistical tools are more generally applied to the quality control of mass-produced products

*CISPR: International Special Committee on Radio Interference

Application of the rule

- Test based on the non-central *t*-distribution
- Rationale thoroughly described in TR CISPR 16-4-3:2007

$$\bar{q} + k \cdot s \leq L$$

$\bar{q} =$ sample mean

$s =$ sample standard deviation

$k =$ factor

$L =$ emission limit

N	3	4	5	6	7	8	9	10	11	12
k	2,04	1,69	1,52	1,42	1,35	1,30	1,27	1,24	1,21	1,20

N is the size of the sample (clause 5.1 of TR CISPR 16-4-3:2007)

The factor *k*

- It is assumed that the production being investigated has a ***normal distribution*** with unknown parameters μ and σ
- Restate the rule as follows:
  - At least the fraction p_1 of the products shall comply with the emission limit with a probability of not less than p_2
- If $F(t\,;\,v,\,\delta)$ is the ***non-central t-CDF*** with $v = N - 1$ DoF and non-centrality parameter δ and $\Phi(z)$ is the normal CDF, then:

$$\delta = k_{p1}\sqrt{\nu + 1} \qquad k_{p1} = \Phi^{-1}(p_1) \qquad F\left(t_{p2}\,;\nu,\delta\right) = p_2$$

$$k = k_{p2} = \frac{t_{p2}}{\sqrt{\nu + 1}}$$

MU and the 80 % / 80 % rule

- In EMC the limit *L* accounts for MU
- If the MU of the test lab is greater than a given MU reference value* then the emission limit is decreased by the difference between the test lab MU and the MU reference value
- Example 1
  - Test lab MU = 4.8 dB
  - MU reference value = 5.4 dB
  - The limit stays
- Example 2
  - Test lab MU = 5.8 dB
  - MU reference value = 5.4 dB
  - The applicable emission limit is $L' = L - 0.4$ dB

* Reference values for the different emission measurement methods are calculated and reported in the standard EN 55016-4-2:2011

The ingredients of our analysis

- What we observe is

$$Q_E = Q + E$$

- $Q \sim N(\mu, \sigma^2)$ is the quality characteristic to be assessed, μ and σ are unknown parameters
- E models an unknown systematic error (the PDF of E is given)
- Q and E are independent each other
- An unobservable value q_i corresponds to each observation q_{Ei}

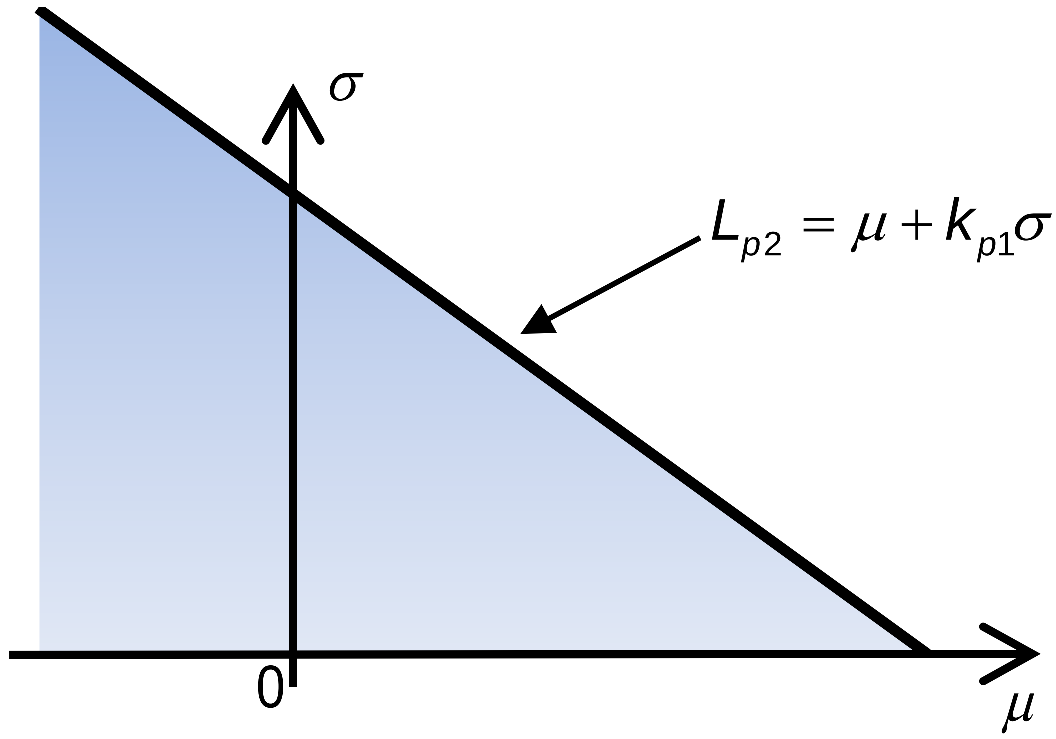
$$q_i = q_{Ei} - e \quad i = 1, 2, \ldots, N$$

- q_i values are independent each other
- q_{Ei} values are correlated by the unknown systematic error e

The recipe – 1st step

Available knowledge → Bayes theorem → Joint density of μ and σ

The recipe – 2nd and final step



- Integration of the joint density of μ and σ over the blue colored area
- The value of L_{p2} such that the volume is p_2 is numerically obtained

Compliance criterion

$$L \geq L_{p2} = \frac{t_{p2}}{\sqrt{N}} s_E + \overline{q_E} = k_{p2} s_E + \overline{q_E}$$

- Same formal expression for the compliance criterion as in the case where MU is neglected, except that now:
  - $k = k_{p2}$ depends on v and p_2, ***and on MU***
  - $\overline{q_E}$ is the sample mean of the ***observations***
  - s_E is the sample standard deviation of the ***observations***

Results – $E \sim N(0, u^2)$

k_{p2}		N											
		2	3	4	5	6	7	8	9	10	20	50	100
$\frac{s_E}{u}$	∞	3.42	2.02	1.67	1.51	1.42	1.35	1.30	1.27	1.24	1.10	0.99	0.95
	10	3.43	2.02	1.68	1.52	1.43	1.36	1.31	1.28	1.25	1.11	1.02	0.98
	3	3.46	2.07	1.75	1.60	1.51	1.45	1.41	1.38	1.35	1.24	1.17	1.15
	1	3.71	2.47	2.18	2.04	1.97	1.91	1.88	1.85	1.83	1.75	1.71	1.69
	0.5	4.50	3.28	2.97	2.84	2.76	2.71	2.68	2.66	2.64	2.58	2.54	2.53
	0.3	5.72	4.40	4.07	3.93	3.86	3.81	3.78	3.76	3.74	3.69	3.66	3.66
	0.2	7.28	5.79	5.45	5.31	5.25	5.20	5.18	5.16	5.14	5.09	5.07	5.07
	0.15	8.81	7.19	6.84	6.70	6.64	6.60	6.58	6.56	6.54	6.49	6.47	6.47
	0.1	11.8	9.99	9.63	9.50	9.44	9.40	9.38	9.36	9.34	9.30	9.27	9.27

$p_1 = p_2 = 0.8$

Results – $E \sim R(-\sqrt{3}u, \sqrt{3}u)$

k_{p2}		N											
		2	3	4	5	6	7	8	9	10	20	50	100
$\frac{s_E}{u}$	∞	3.42	2.02	1.67	1.51	1.42	1.35	1.30	1.27	1.24	1.10	0.99	0.95
	10	3.42	2.02	1.68	1.52	1.43	1.36	1.31	1.28	1.25	1.11	1.02	0.98
	3	3.45	2.07	1.75	1.60	1.51	1.46	1.41	1.38	1.36	1.26	1.21	1.20
	1	3.68	2.48	2.23	2.12	2.06	2.02	1.99	1.98	1.96	1.92	1.89	1.89
	0.5	4.40	3.45	3.23	3.13	3.08	3.05	3.03	3.01	3.00	2.96	2.93	2.93
	0.3	5.81	4.87	4.62	4.52	4.47	4.43	4.41	4.40	4.39	4.34	4.32	4.31
	0.2	7.73	6.63	6.36	6.25	6.20	6.17	6.14	6.13	6.12	6.07	6.05	6.05
	0.15	9.62	8.37	8.09	7.98	7.93	7.90	7.88	7.86	7.85	7.80	7.78	7.78
	0.1	13.3	11.9	11.6	11.4	11.4	11.4	11.3	11.3	11.3	11.3	11.2	11.2

$p_1 = p_2 = 0.8$

Some details about the analysis

- Two independent derivations (same results):
  - Q_i and E are normal and independent then a multivariate normal distribution is considered as the likelihood function

$$f\left(\mu,\sigma \mid \mathbf{q_E}\right) \propto l\left(\mathbf{q_E} \mid \mu,\sigma\right)\cdot f\left(\mu,\sigma\right) = N\left(\mathbf{q_E};\boldsymbol{\mu},\boldsymbol{\Sigma}\right)\cdot f\left(\mu,\sigma\right)$$

$$\boldsymbol{\mu} = \left[\mu,\mu,\ldots,\mu\right]^T \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2+u^2 & \ldots & u^2 \\ \vdots & \ddots & \vdots \\ u^2 & \cdots & \sigma^2+u^2 \end{bmatrix}$$

  - Application of Bayes theorem with use of model prior: more slippery* but also more general (Q_i and E may be non normal)

$$f\left(\mu,\sigma \mid \mathbf{q_E}\right) \propto \int_{-\infty}^{\infty} \prod_{i=1}^{N} N\left(q_{Ei};\mu+e,\sigma^2\right)\cdot f\left(e\right)de\cdot f\left(\mu,\sigma\right)$$

* Generating very long discussions between me and Francesca

Summary and conclusion

- Analytical expressions for the joint density of the parameters of a normal mass-produced production were found
- Correlation among observations due to the systematic measurement error is taken into account
- The usual improper priors were assigned to the parameters
- In the limit were MU is negligible with respect to the sample standard deviation the new results and the consolidated ones (non-central *t*-CDF) are the same
- The analysis is valid if the non-repeatability of the measuring instrumentation is negligible with respect to the variability of the production process

The “slippery” derivation

$$f\left(\mu,\sigma \mid \mathbf{q_E},\mathbf{q},e\right) \propto f\left(\mathbf{q_E},\mathbf{q},e \mid \mu,\sigma\right)\cdot f\left(\mu,\sigma\right)$$

$$f\left(\mathbf{q_E},\mathbf{q},e \mid \mu,\sigma\right) \propto f\left(\mathbf{q_E} \mid \mathbf{q},e,\mu,\sigma\right)\cdot f\left(\mathbf{q},e \mid \mu,\sigma\right)$$

$$f\left(\mathbf{q_E} \mid \mathbf{q},e,\mu,\sigma\right) = f\left(\mathbf{q_E} \mid \mathbf{q},e\right) = \prod_{i=1}^{N}\delta\left(q_{Ei}-q_i-e\right)$$

$$f\left(\mathbf{q},e \mid \mu,\sigma\right) = f\left(\mathbf{q} \mid \mu,\sigma\right)\cdot f\left(e\right)$$

$$f\left(\mathbf{q} \mid \mu,\sigma\right) = \prod_{i=1}^{N} f\left(q_i \mid \mu,\sigma\right) = \prod_{i=1}^{N} N\left(q_i;\mu,\sigma^2\right)$$

$$f\left(\mu,\sigma \mid \mathbf{q_E},\mathbf{q},e\right) \propto \prod_{i=1}^{N}\delta\left(q_{Ei}-q_i-e\right)\cdot\prod_{i=1}^{N} N\left(q_i;\mu,\sigma^2\right)\cdot f\left(e\right)\cdot f\left(\mu,\sigma\right)$$

$$f\left(\mu,\sigma \mid \mathbf{q_E},e\right) \propto \prod_{i=1}^{n} N\left(q_{Ei}-e;\mu,\sigma^2\right)\cdot f\left(e\right)\cdot f\left(\mu,\sigma\right) = \prod_{i=1}^{N} N\left(q_{Ei};\mu+e,\sigma^2\right)\cdot f\left(e\right)\cdot f\left(\mu,\sigma\right)$$

$$f\left(\mu,\sigma \mid \mathbf{q_E}\right) = \int_{-\infty}^{\infty}\prod_{i=1}^{N} N\left(q_{Ei};\mu+e,\sigma^2\right)\cdot f\left(e\right)de\cdot f\left(\mu,\sigma\right)$$