Determination of KCRV, u(KCRV) & DoE

Final Proposal by KCWG to CCRI(II)

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CCRI(II) Meeting, BIPM,
14-16 May 2013

Estimators for mean

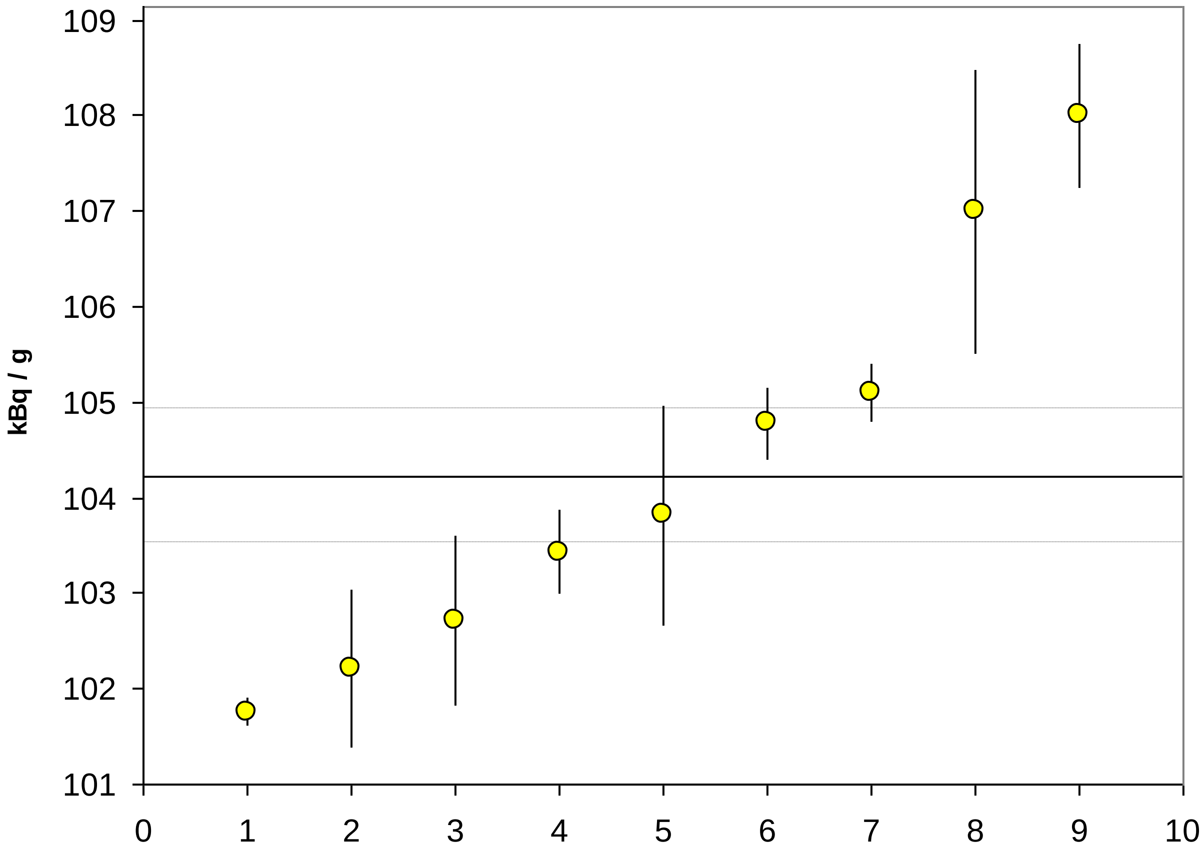
arithmetic
weighted
Mandel-Paule
Power Moderated Mean

Arithmetic mean

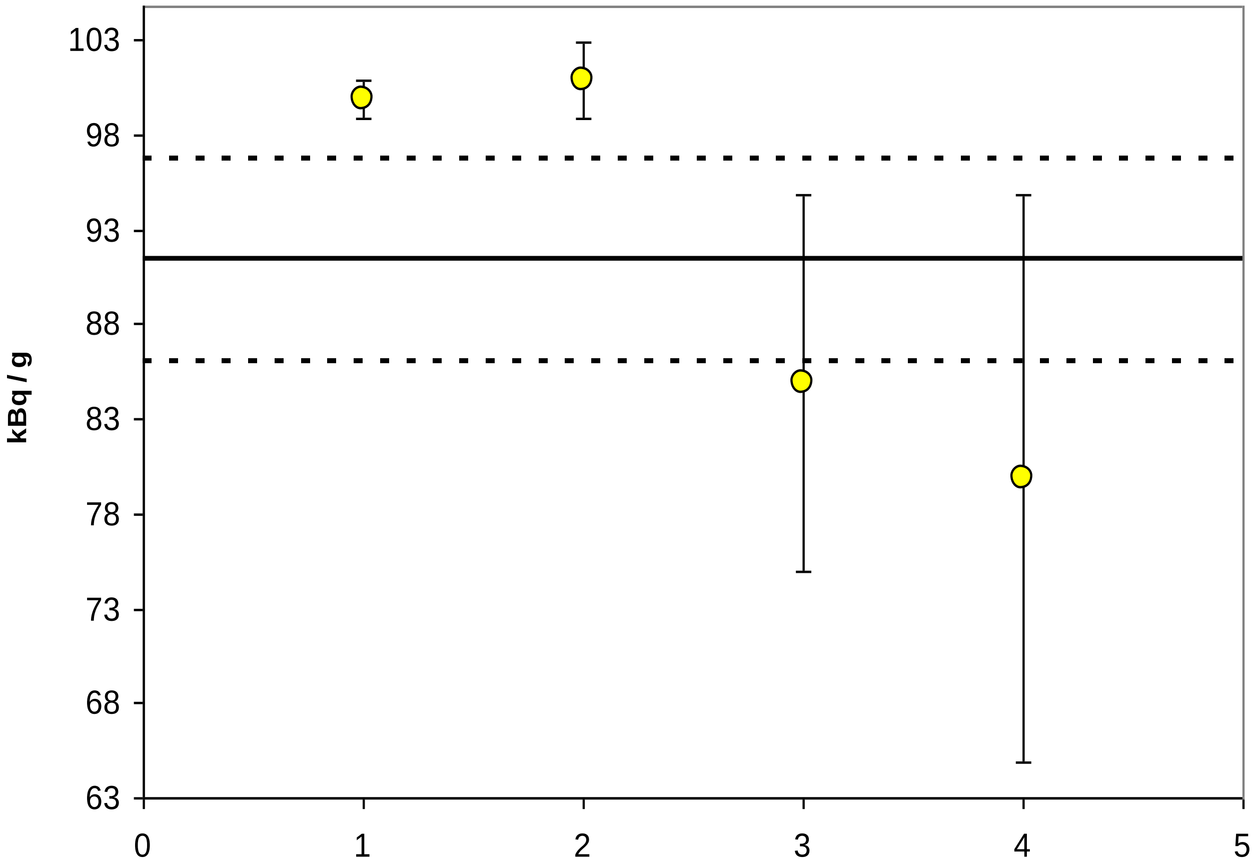
$$x_{\text{ref}} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$u^2(x_{\text{ref}}) = \sum_{i=1}^{N}(x_i - x_{\text{ref}})^2 / (N-1).$$

Good: ignore wrong unc



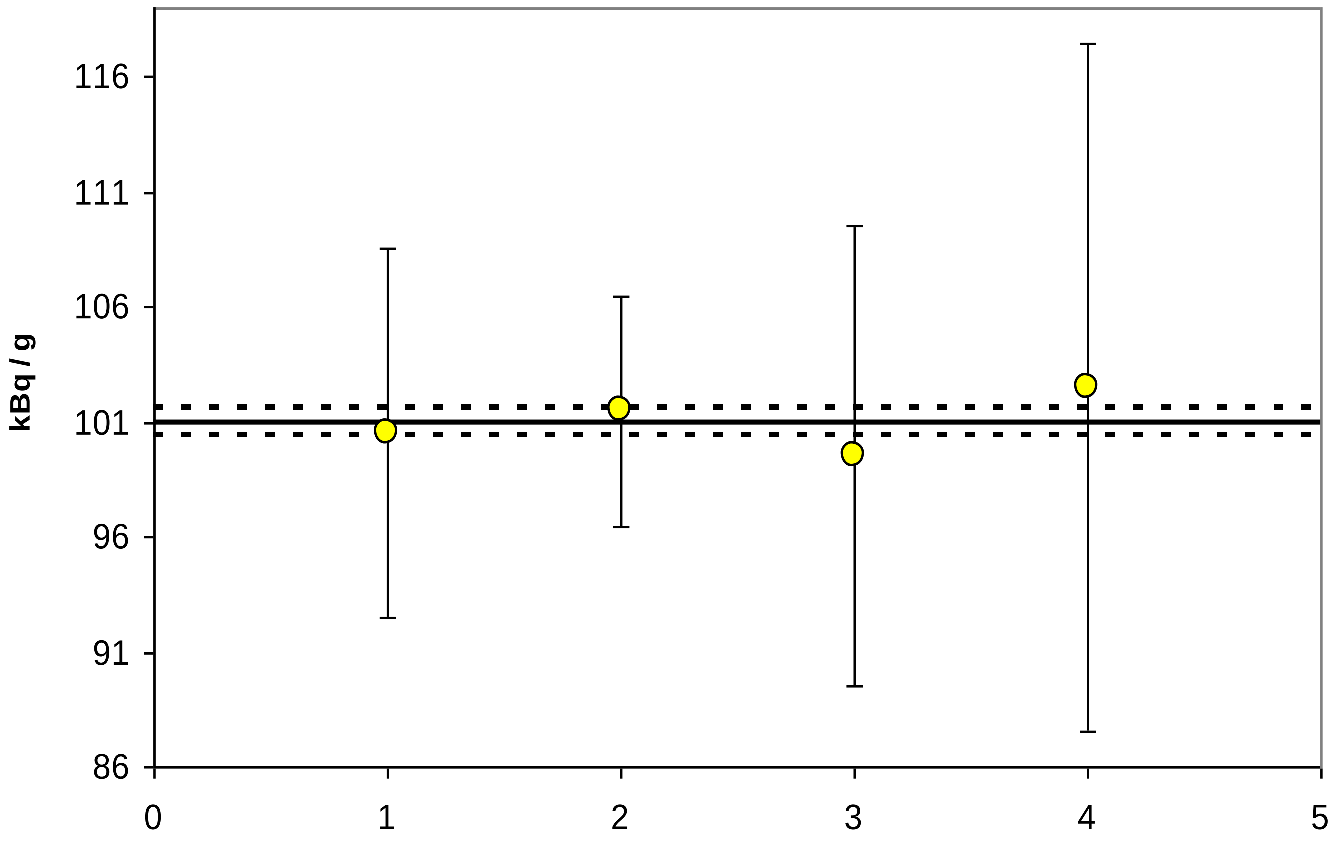
Bad: inefficient



biased mean

large uncertainty
= not "efficient"

Bad: low sample variance



low uncertainty

Remedy against low unc

Calculate uncertainty from maximum of

- propagated sum of stated uncertainties
- sample variance

$$u(\bar{x}) = \max\left(\frac{1}{N}\sqrt{\sum_{i=1}^{N} u_i^2}, \sqrt{\sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{N(N-1)}} \right)$$

Arithmetic mean

Best solution is close to *arithmetic mean* if

- measurement uncertainty contains no useful information
- magnitude error on uncertainty is >2x larger than magnitude due to metrological reasons

= the “best” with “bad uncertainty data”

= inefficient with “consistent data”

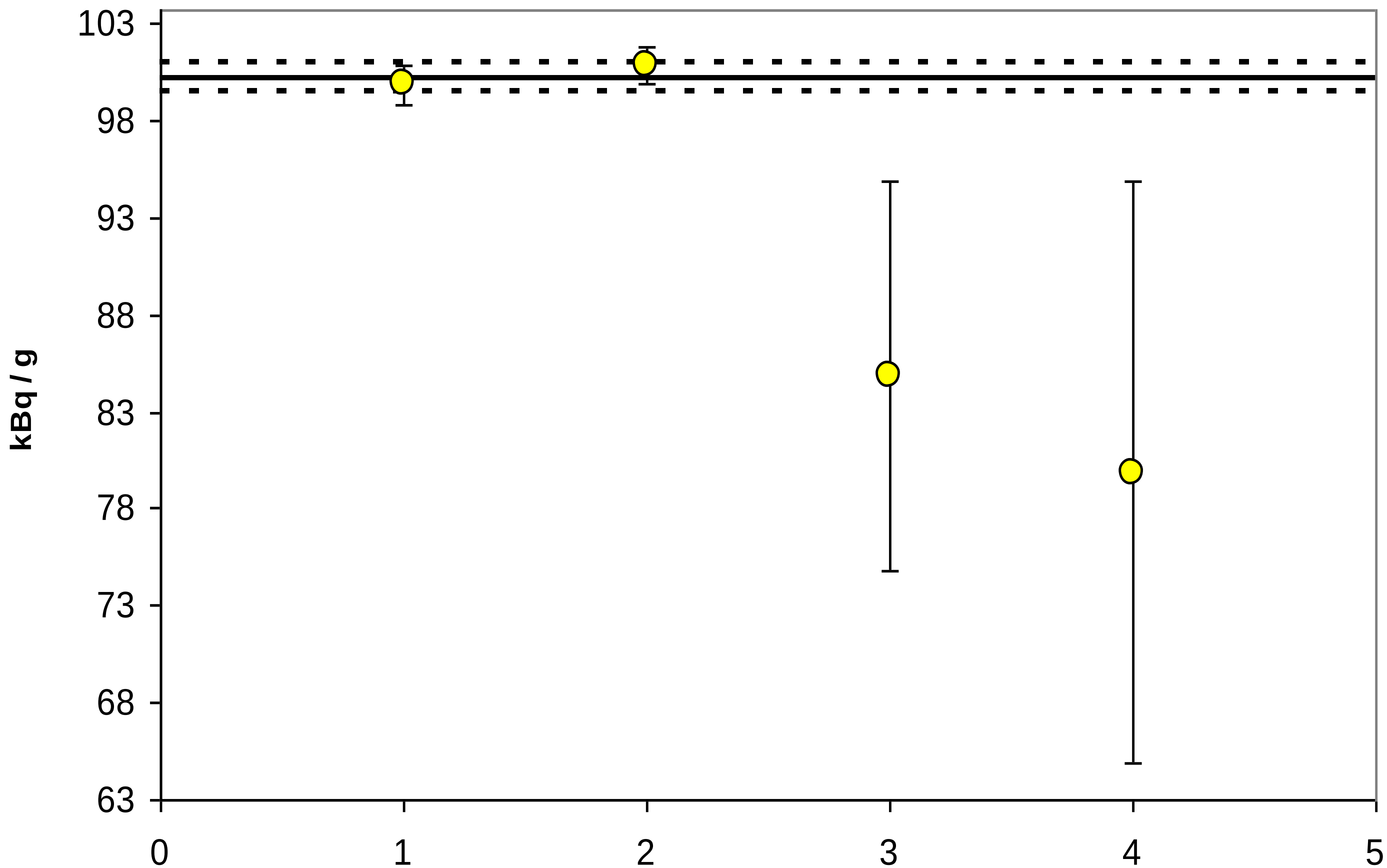
Weighted mean

$$x_{\text{ref}} = u^2(x_{\text{ref}}) \sum_{i=1}^{N} \frac{x_i}{u_i^2}$$

$$\frac{1}{u^2(x_{\text{ref}})} = \sum_{i=1}^{N} \frac{1}{u_i^2}$$

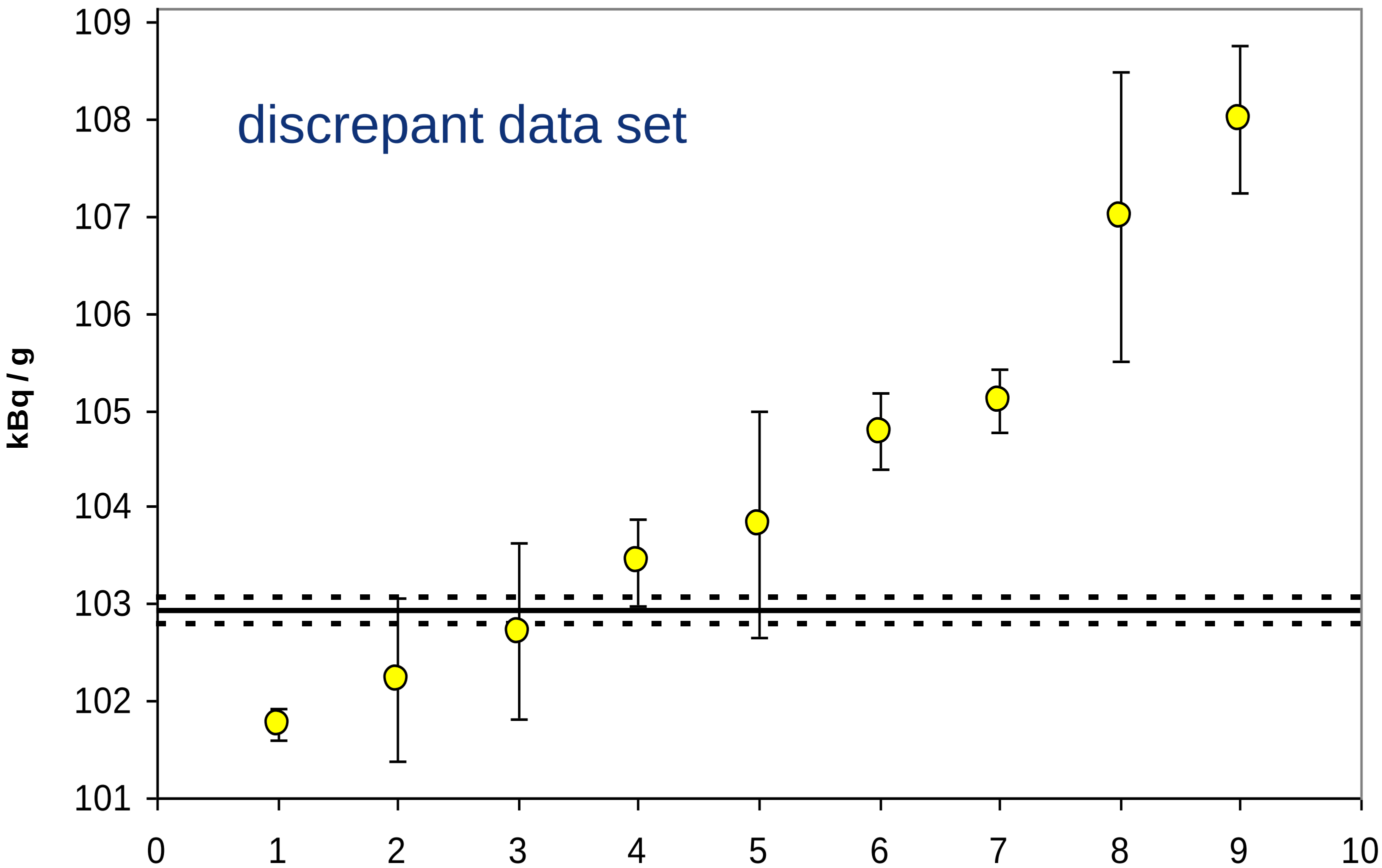
statistical weight = reciprocal variance associated with x_i

Good: efficient



efficient

Bad: sensitive to error



biased mean
low uncertainty

Weighted mean

European Commission

Best solution is close to *weighted mean* if

- measurement uncertainties are correct
- ‘value’ and ‘uncertainty’ outliers are excluded

= the “best” with “perfect data”

= sensitive to “low uncertainty” outliers

Mandel-Paule mean

$$x_{\text{ref}} = u^2(x_{\text{ref}}) \sum_{i=1}^{N} \frac{x_i}{u_i^2 + s^2},$$

$$\frac{1}{u^2(x_{\text{ref}})} = \sum_{i=1}^{N} \frac{1}{u_i^2 + s^2},$$

s^2 is artificially added “interlaboratory” variance to make reduced chi = 1

$$\tilde{\chi}_{\text{obs}} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} \frac{(x_i - x_{\text{ref}})^2}{u_i^2}}.$$

Mandel-Paule mean

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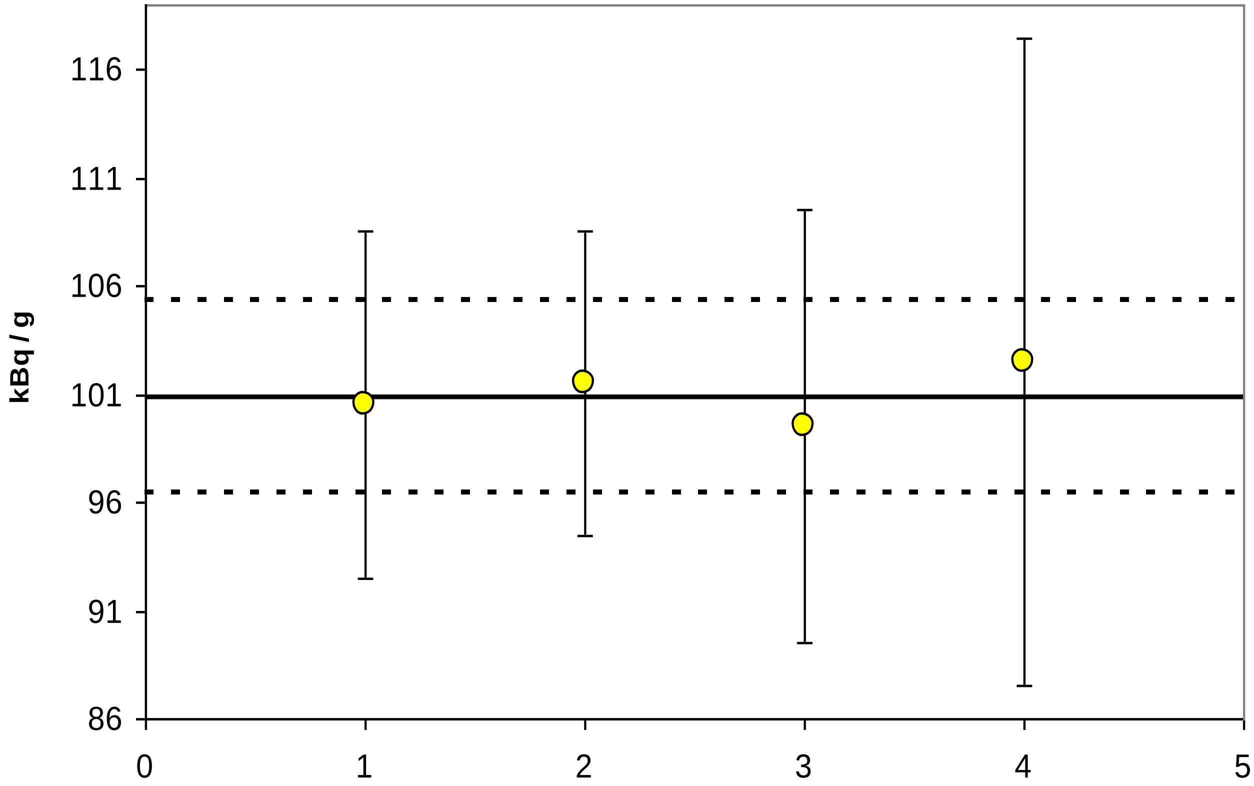
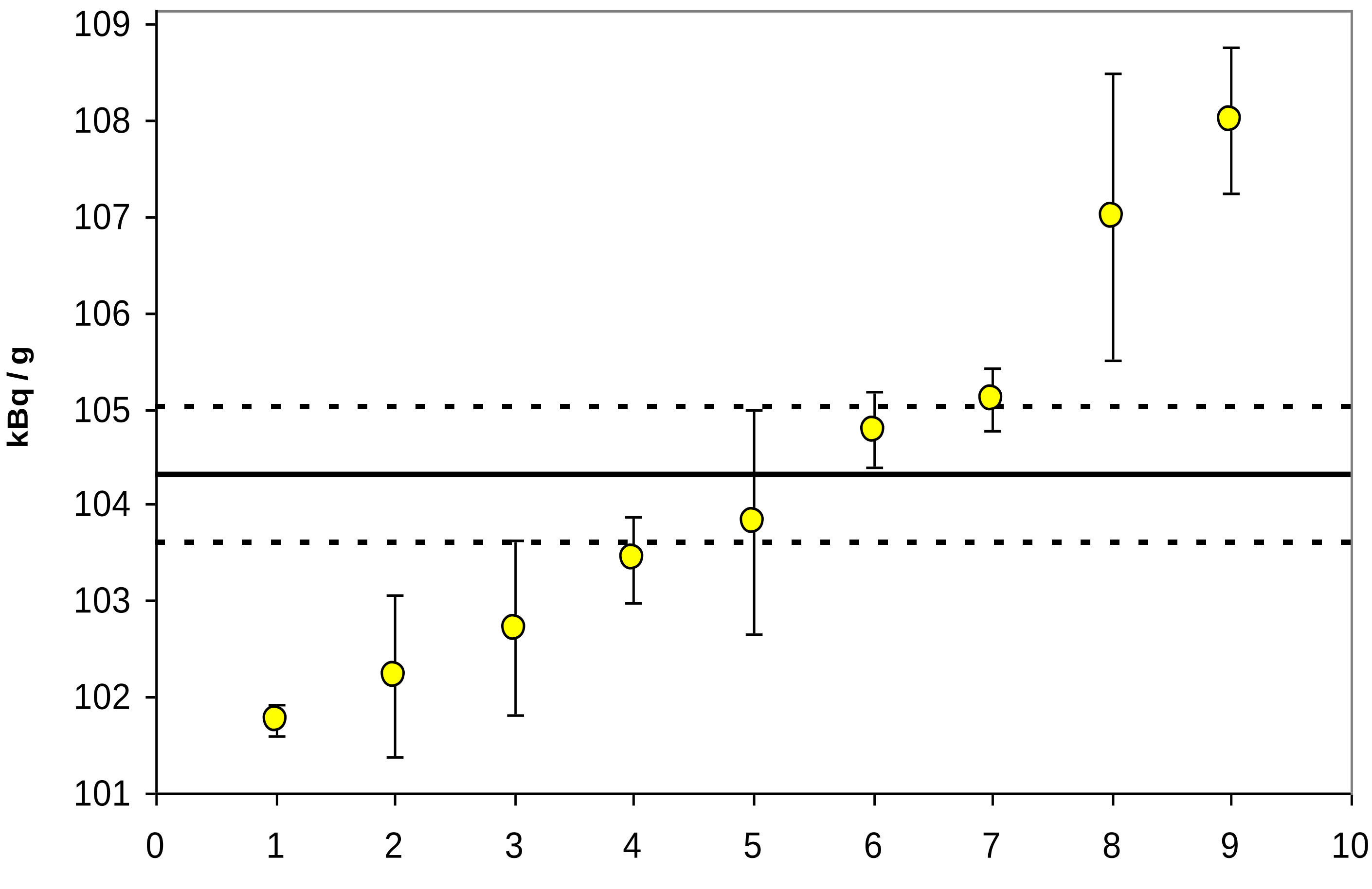
= weighted mean for a consistent data set



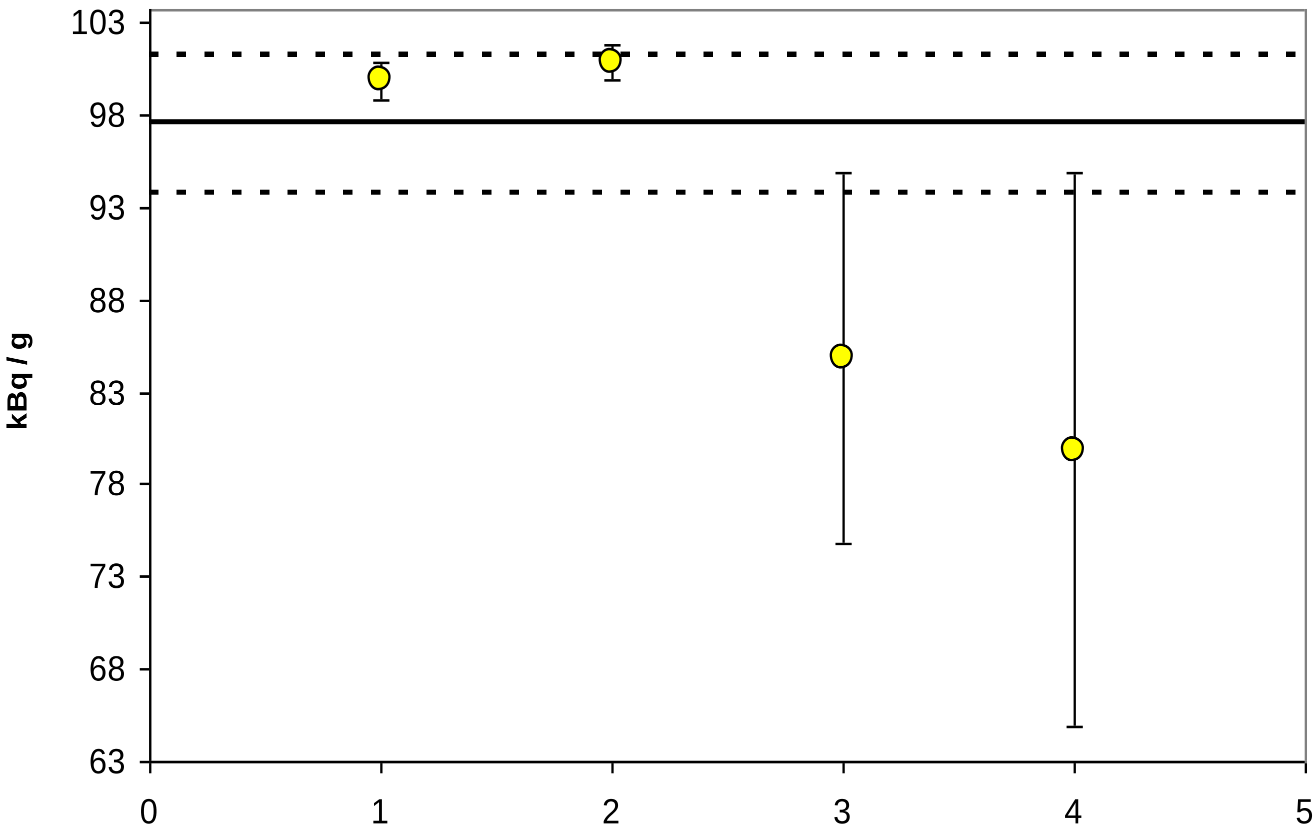
Mandel-Paule mean

≈ arithmetic mean for an extremely discrepant data set



Mandel-Paule mean

= intermediate between arithmetic and weighted mean for a slightly discrepant data set



Mandel-Paule mean

Best solution is close to Mandel-Paule mean if

- measurement uncertainties are informative
- ‘value’ and ‘uncertainty’ outliers are symmetric

= one of the “best” with “imperfect data”
if no tendency to underestimate uncertainty

Power Moderated Mean - PMM

$$x_{\mathrm{ref}} = \sum_{i=1}^{N} w_i x_i$$

$$w_i = u^2(x_{\mathrm{ref}})\left[\left(\sqrt{u_i^{\,2} + s^2}\right)^{\alpha} S^{2-\alpha}\right]^{-1}$$

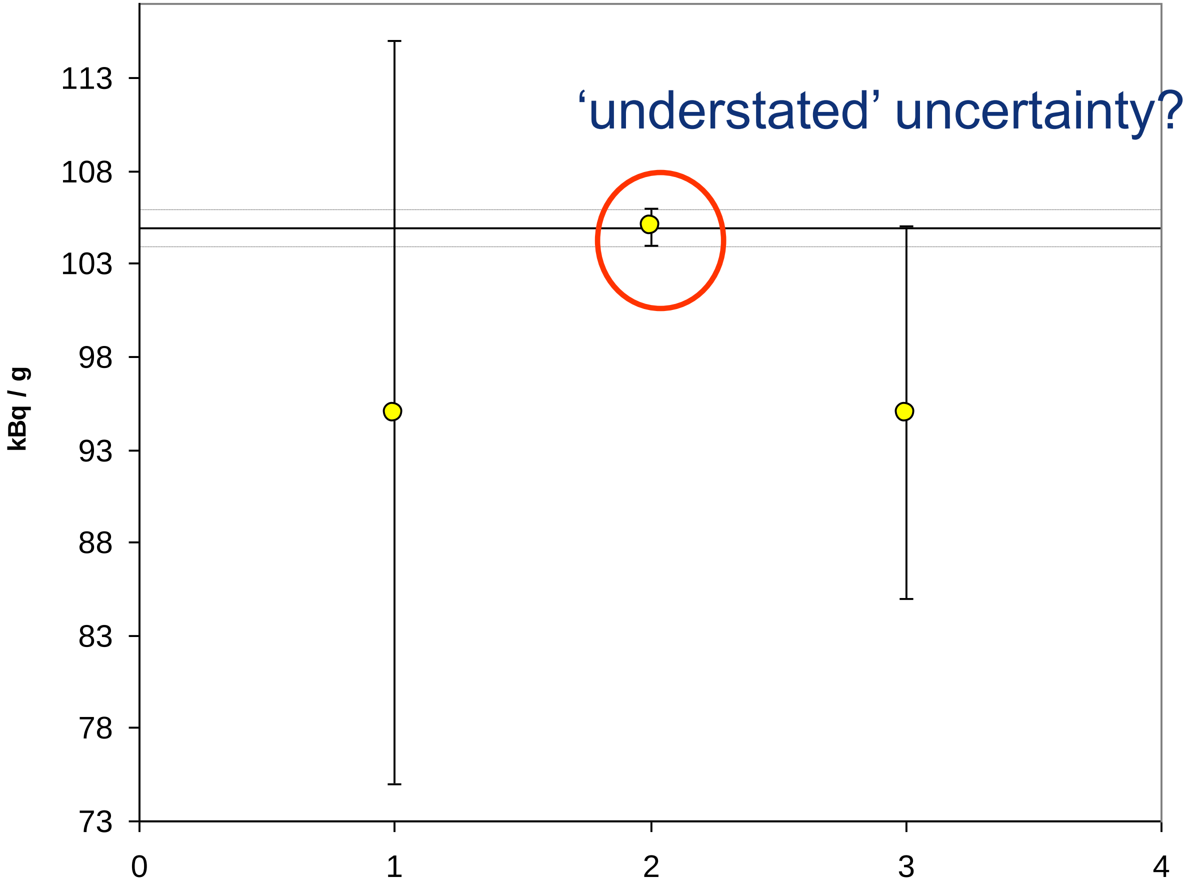
$$S = \sqrt{N \cdot \max\left(u^2(\bar{x}), u^2(x_{\mathrm{mp}})\right)}$$

S is a typical uncertainty per datum (max arithmetic or M-P unc)

$0<\alpha<2$ = power reflects level of trust in uncertainties

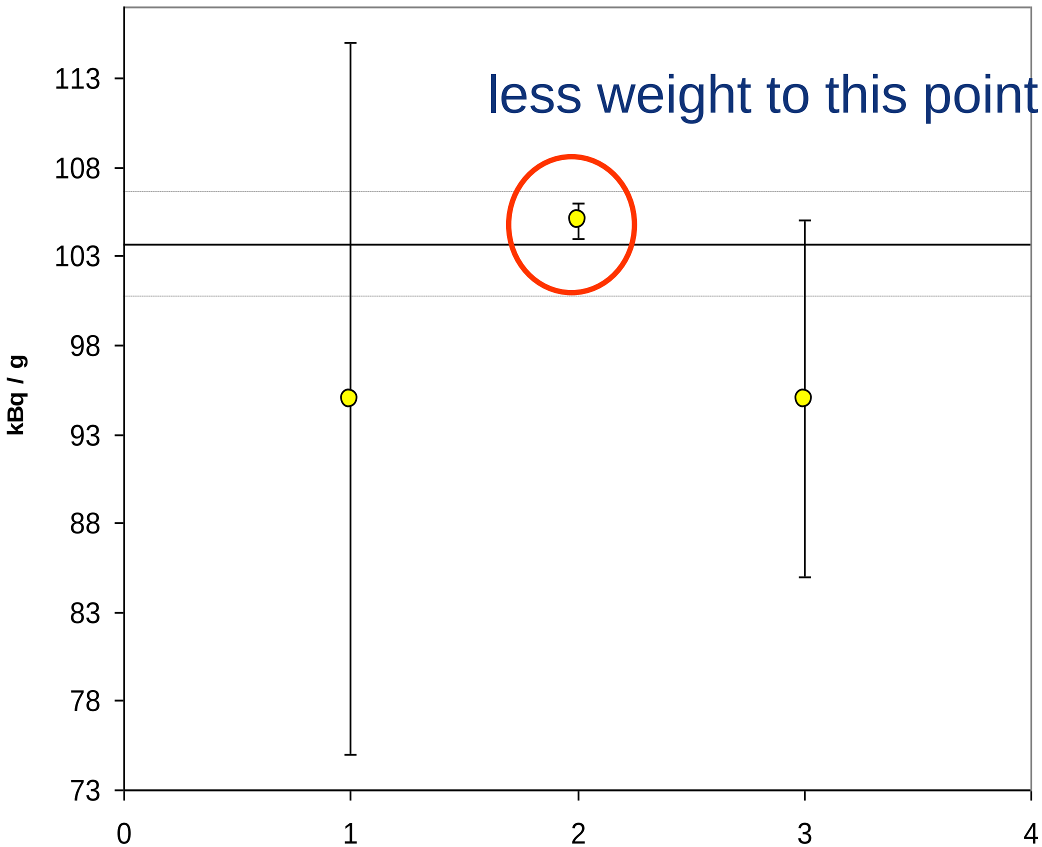
PMM: α=2

= Mandel-Paule mean



PMM: $\alpha=1$

= closer to arithmetic mean



Choice of power α

power	reliability of uncertainties
$\alpha = 0$	uninformative uncertainties (arithmetic mean)
$\alpha = 0$	uncertainty variation due to error at least twice the variation due to metrological reasons (arithmetic mean)
$\alpha = 2\text{-}3/N$	informative uncertainties with a tendency of being rather underestimated than overestimated (intermediately weighted mean)
$\alpha = 2$	informative uncertainties with a modest error; no specific trend of underestimation (Mandel-Paule mean)
$\alpha = 2$	accurately known uncertainties, consistent data (weighted mean)

Test of Estimators by Computer Simulation

arithmetic
weighted
Mandel-Paule
Power Moderated Mean

Efficiency for discrepant data

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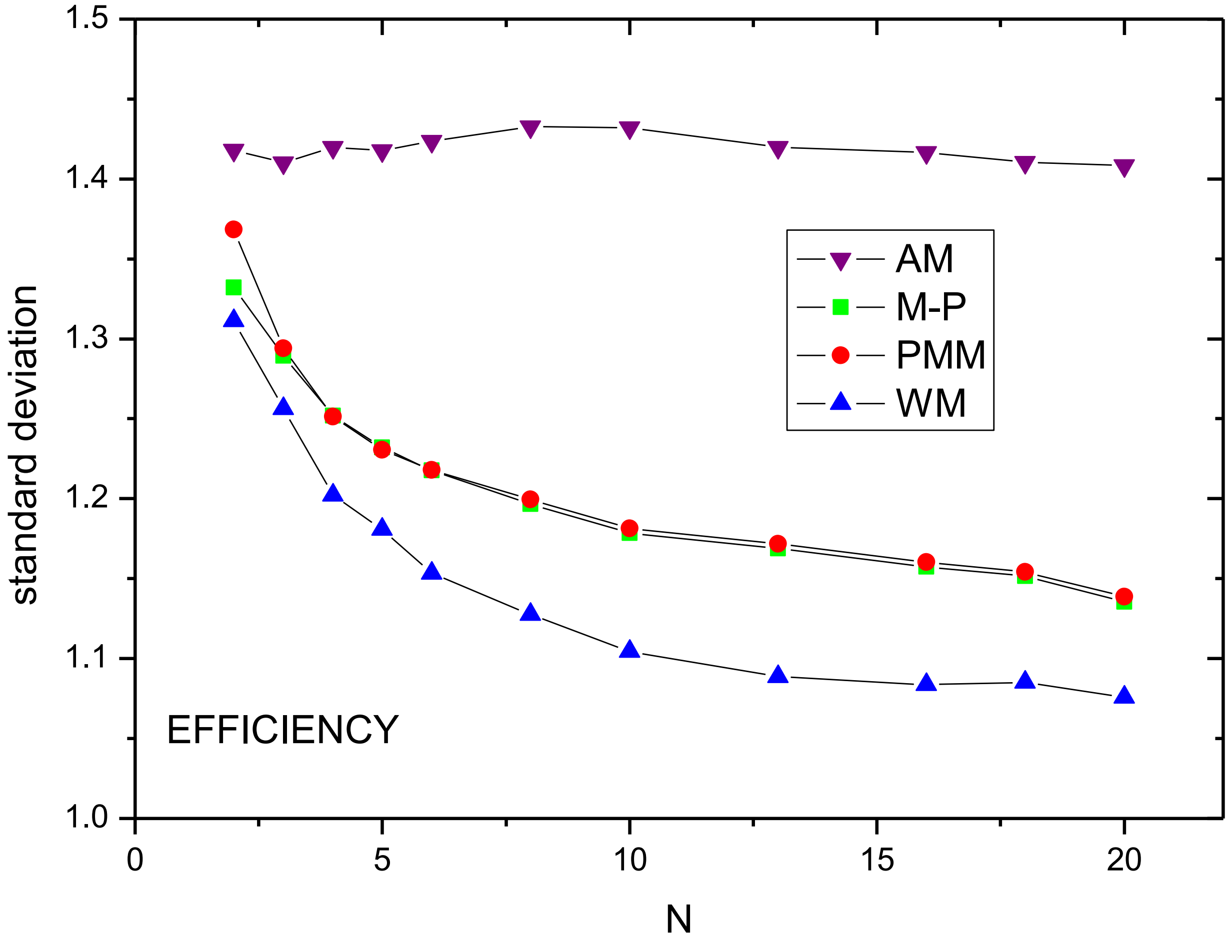
arithmetic < PMM, M-P < weighted



Reliability of uncertainty

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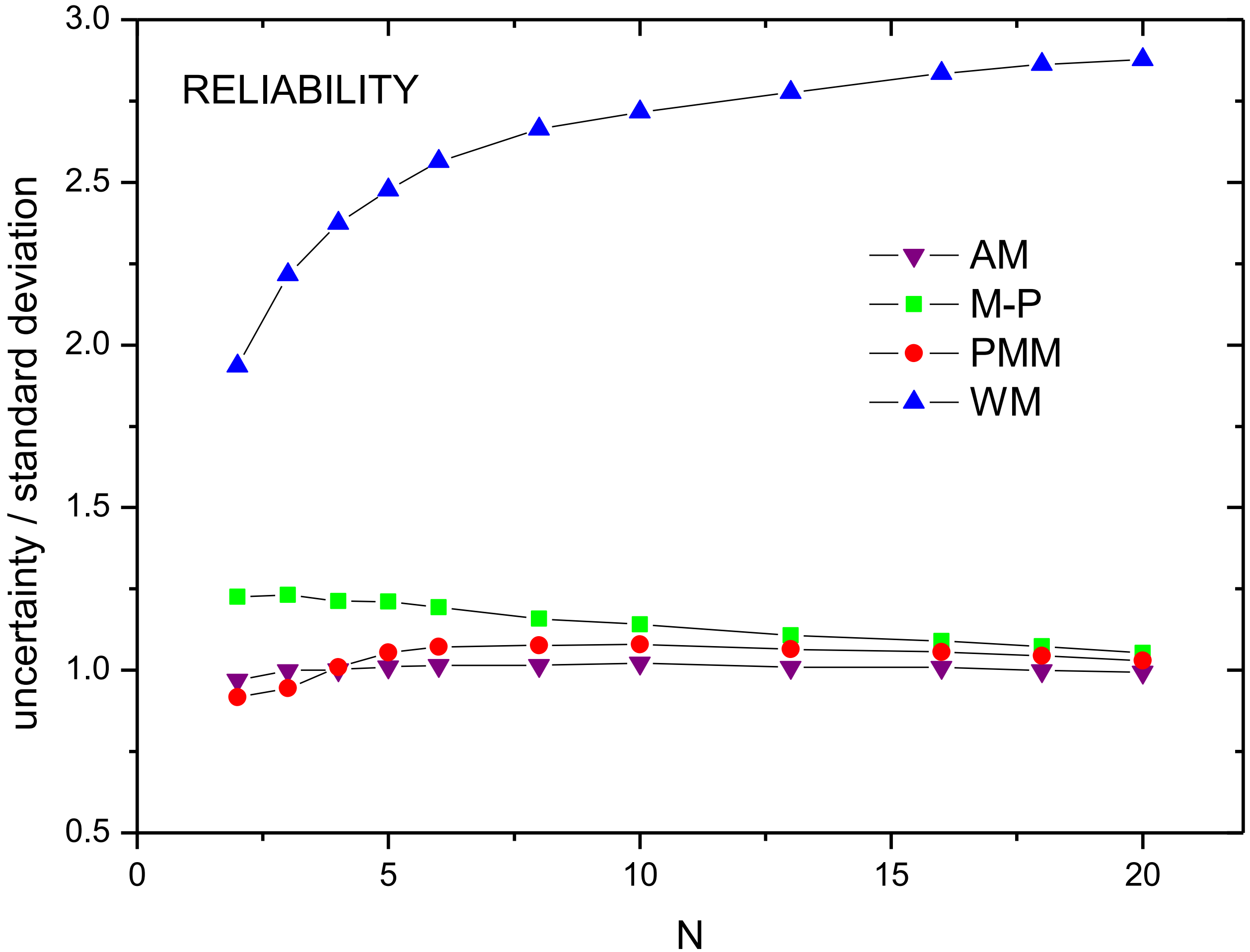
weighted << M-P < PMM < arithmetic



Power Moderated Mean

Best solution is close to PMM if

- measurement uncertainties are informative
- uncertainties tend to be understated
- data seem consistent but are not

= one of the “very best” with “imperfect data”

= more realistic uncertainty than Mandel-Paule mean

= more adjustable to quality of data than M-P mean

Outlier identification

generally applicable method

Outlier identification

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CCRI(II) is the final arbiter regarding correcting or excluding any data from the calculation of the KCRV.

Statistical tools may be used to indicate data that are extreme.

= a way to protect the KCRV against erroneous data, data with understated uncertainty, extreme data asymmetrically disposed to the KRCV

= a way to lower the uncertainty on the KCRV

Outlier identification

$$|e_i| > ku(e_i), \quad e_i = x_i - x_{\text{ref}}$$

$$u^2(e_i) = u^2(x_{\text{ref}})(\frac{1}{w_i} - 1)$$

x_i included in mean

$$u^2(e_i) = u^2(x_{\text{ref}})(\frac{1}{w_i} + 1)$$

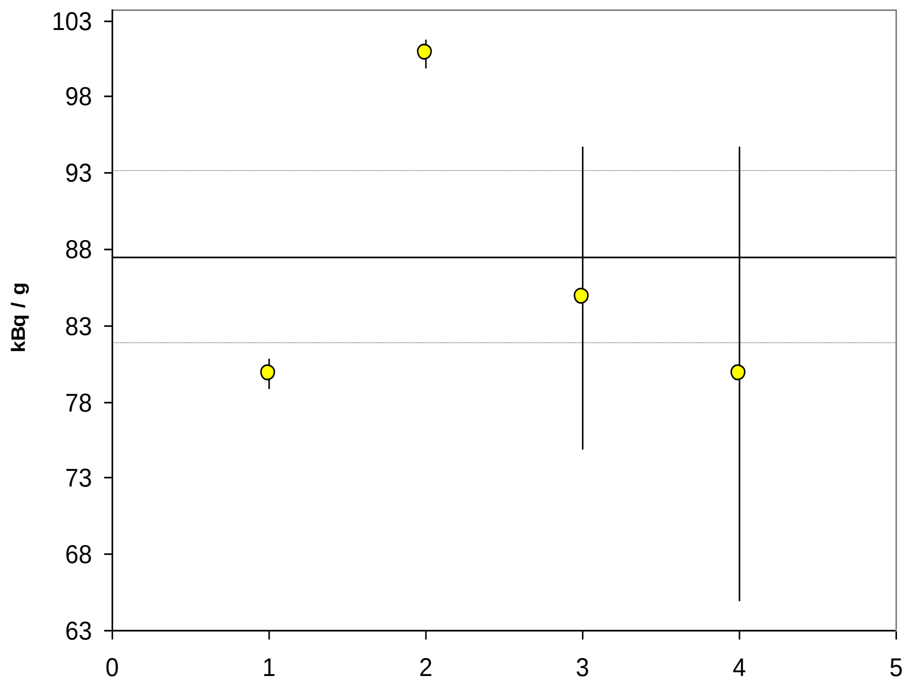
x_i excluded from mean

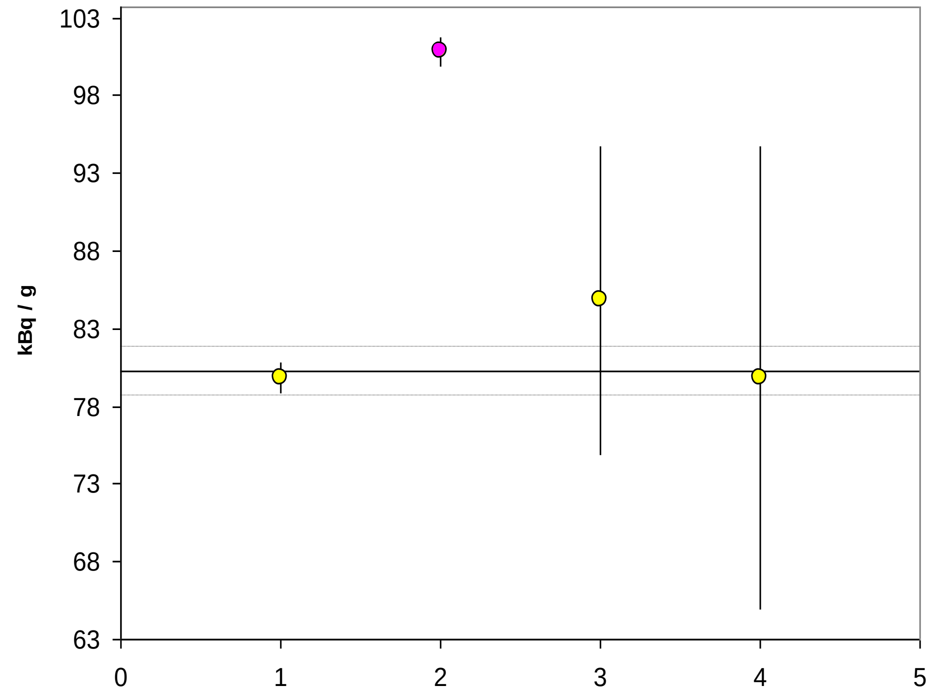
- valid for any type of mean, using normalised w_i
- default $k = 2.5$

Outlier or not?

Both options are possible within the method.

→ outlier rejection should be based on technical grounds





Degree of equivalence

generally applicable method

Degree of equivalence

$$d_i = x_i - x_{\mathrm{ref}}, \quad U(d_i) = 2u(d_i)$$

$$u^2(d_i) = (1-2w_i)u_i^2 + u^2(x_{\mathrm{ref}})$$ x_{i} included in mean

$$u^2(d_i) = u_i^2 + u^2(x_{\mathrm{ref}})$$ x_{i} excluded from mean

- valid for any type of mean

Conclusions

The **Power Moderated Mean** keeps a fine balance between efficiency and robustness, while providing also a reliable uncertainty.

Outlier identification and degrees of equivalence are readily obtained.